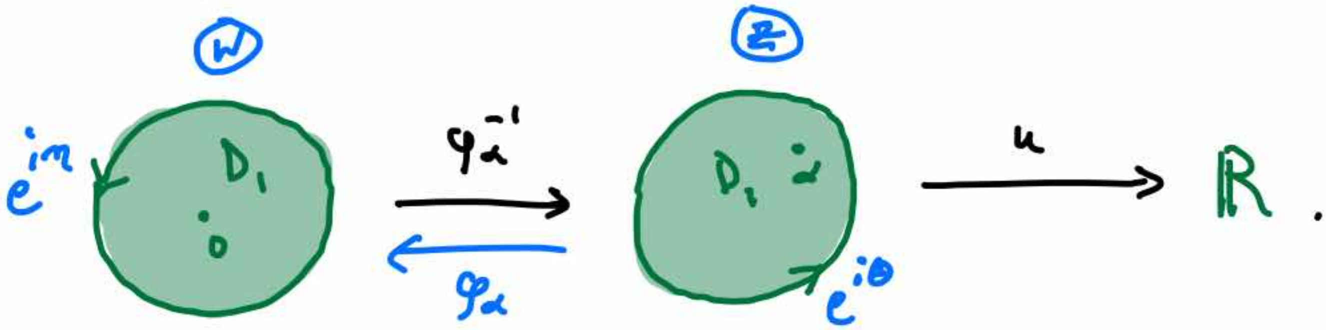
Lecture 31 : Poisson formula

I. Harmonic functions on D_1

So what is $u(\alpha)$ if $\alpha \in D(z_0, r)$ <u>isn't</u> the center of the disk? For simplicity set $z_0 = 0$, $r = 1$, † and consider $u \in \mathcal{H}(\overline{D_1})$ (i.e. u is harmonic on an open set $U \supset \overline{D_1}$). Recall $\varphi_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha} z}$ and consider the composition



Note that

$$\varphi_\alpha^{-1} \text{ holo.} \implies u \circ \varphi_\alpha^{-1} \text{ harmonic} \underset{\text{MVT}}{\implies}$$

$$u(\alpha) = (u \circ \varphi_\alpha^{-1})(0) = \frac{1}{2\pi}\int_0^{2\pi} u(\underbrace{\varphi_\alpha^{-1}(\overbrace{e^{i\eta}}^{w})}_{e^{i\theta}})\, \underbrace{d\eta}_{\substack{d\arg(w) \\ = d\arg(\varphi_\alpha(z))}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, d\arg \varphi_\alpha(z).$$

† look in Ahlfors for the general formula for an R-disk.

Since $d\arg\varphi_\alpha(z) \overset{\varphi_\alpha(z)\text{ on unit circle}}{=} -i\,d\log\varphi_\alpha(z) = -i\dfrac{\varphi_\alpha'(z)}{\varphi_\alpha(z)}\,dz$

$$= -i\,\frac{(1-|\alpha|^2)/(1-\bar\alpha z)^{\cancel{2}}}{(z-\alpha)\,\cancel{/(1-\bar\alpha z)}}\,dz \underset{z\bar z=1}{=} \frac{1-|\alpha|^2}{(z-\alpha)(\bar z-\bar\alpha)}\,(-i\,\bar z\,dz)$$

$$= \frac{1-|\alpha|^2}{|z-\alpha|^2}\,d\theta\,, \quad \text{we obtain}$$

(1) $$\boxed{u(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,\frac{1-|\alpha|^2}{|e^{i\theta}-\alpha|^2}\,d\theta.}$$

Noting that (for $z=e^{i\theta}$)

$$\frac{z+\alpha}{z-\alpha} = \frac{(z+\alpha)(\bar z-\bar\alpha)}{|z-\alpha|^2} = \frac{1-|\alpha|^2+i(\cdots)}{|z-\alpha|^2}\,, \quad \text{this becomes}$$

(2) $$\boxed{u(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,\mathrm{Re}\left\{\frac{e^{i\theta}+\alpha}{e^{i\theta}-\alpha}\right\}d\theta.}$$

Finally, writing $\alpha = ae^{i\psi}$ and

$$|e^{i\theta}-ae^{i\psi}|^2 = (e^{i\theta}-ae^{i\psi})(e^{-i\theta}-ae^{-i\psi})$$
$$= 1-a\left\{e^{i(\psi-\theta)}+e^{i(\theta-\psi)}\right\}+a^2 \quad \text{yields}$$

(3) $$\boxed{u(ae^{i\psi}) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\underbrace{\frac{1-a^2}{1-2a\cos(\theta-\psi)+a^2}}_{P_a(\theta-\psi)}\,d\theta,}$$

where $P_a(\theta-\psi)$ is the Poisson kernel. Formulas

(1)–(3) are all versions of the POISSON FORMULA.

The Poisson kernel takes the appealing form

$$P_r(\mu) := \mathrm{Re}\left(\frac{e^{i\mu}+r}{e^{i\mu}-r}\right) = \mathrm{Re}\left(\frac{1+e^{-i\mu}r}{1-e^{-i\mu}r}\right) = \mathrm{Re}\left(\frac{1+e^{i\mu}r}{1-e^{i\mu}r}\right)$$

conjugate

$$= \mathrm{Re}\left\{(1+re^{i\mu})(1+re^{i\mu}+r^2e^{2i\mu}+\cdots)\right\}$$

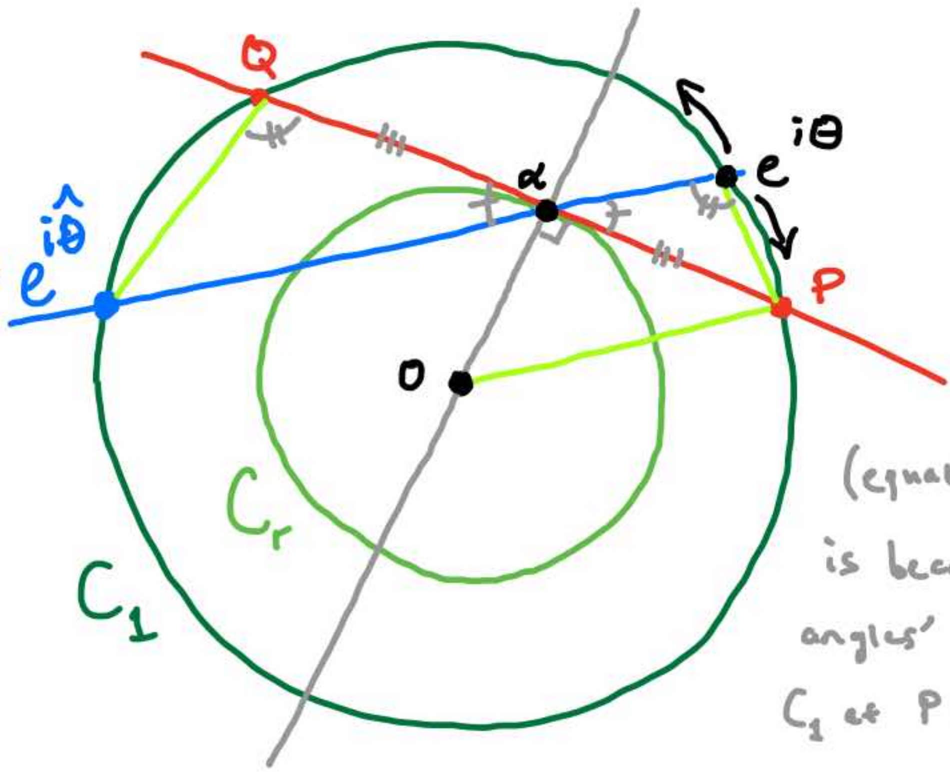
$$= \mathrm{Re}\left\{1+2\sum_{n\geq 1} r^n e^{in\mu}\right\}$$

$$= 1+2\sum_{n\geq 1} r^n \cos(n\mu)$$

$$= \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\mu}.$$

But what does it <u>mean</u>? Consider the picture †

by defn. 2nd intersection of $L_{\alpha,e^{i\theta}}$ with the unit circle C_1 ⇝ $e^{i\hat{\theta}}$



(equality of $\measuredangle$'s is because both angles' rays hit C_1 at P & $e^{i\hat{\theta}}$)

† the equality of edge lengths is NOT meant to be an ingredient in the similarity of triangles on the next page!

Clearly

- $\Delta P\alpha O$ right triangle $\Rightarrow$ $|\overleftrightarrow{P\alpha}| = |\overleftrightarrow{\alpha Q}| = \sqrt{1-|\alpha|^2}$
- $\Delta \alpha e^{i\theta} P \sim \Delta \alpha Q e^{i\hat\theta}$ $\Rightarrow$ $\dfrac{\sqrt{1-|\alpha|^2}}{|e^{i\hat\theta}-\alpha|} = \dfrac{|e^{i\theta}-\alpha|}{\sqrt{1-|\alpha|^2}}$

$$\Rightarrow \underline{1-|\alpha|^2 = |(e^{i\theta}-\alpha)(e^{i\hat\theta}-\alpha)|} \quad (\#)$$

- $\arg(e^{i\theta}-\alpha) \equiv \pi + \arg(e^{i\hat\theta}-\alpha) \Rightarrow \pi = \arg(e^{i\theta}-\alpha) \underbrace{- \arg(e^{i\hat\theta}-\alpha)}_{+\arg(e^{-i\hat\theta}-\bar\alpha)}$

$$\overset{\text{by }(\#)}{\Longrightarrow} \underline{-(e^{i\theta}-\alpha)(e^{-i\hat\theta}-\bar\alpha) = 1-|\alpha|^2} \quad (*)$$

Taking dlog of (minus) both sides of (*) gives

- $d\log(e^{i\theta}-\alpha) = -d\log(e^{-i\hat\theta}-\bar\alpha) \Rightarrow \dfrac{e^{i\theta}\,d\theta}{e^{i\theta}-\alpha} = \dfrac{e^{-i\hat\theta}\,d\hat\theta}{e^{-i\hat\theta}-\bar\alpha}$

$$\Rightarrow \frac{d\hat\theta}{d\theta} = e^{i(\theta+\hat\theta)}\cdot\frac{e^{-i\hat\theta}-\bar\alpha}{e^{i\theta}-\alpha} \overset{\mathbb{R}}{=} \left|\frac{e^{i\hat\theta}-\alpha}{e^{i\theta}-\alpha}\right|$$

$$\overset{\text{by }(\#)}{\Longrightarrow} \frac{d\hat\theta}{d\theta} = \frac{1-|\alpha|^2}{|e^{i\theta}-\alpha|^2}.$$

Substituting this into (1) gives $u(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\hat\theta$,

hence by symmetry of the correspondence $\theta \Leftrightarrow \hat\theta$

$$(4) \qquad \boxed{u(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\hat\theta})\,d\theta}$$

the "geometric interpretation" of Poisson's formula.

Now, we show how to weaken the hypotheses a bit.

Theorem If $u \in C^0(\overline{D}_1)$ with $u|_{D_1}$ harmonic, then
$$u(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, P_a(\theta-\psi)\, d\theta$$
for all $\alpha = ae^{i\psi} \in D_1$.

Proof: Let $r \in (0,1)$, so that $u \circ \mu_r \in \mathcal{H}(\overline{D}_1)$. multiplication by r

Poisson $\Rightarrow$ $u(r\alpha) = (u\circ\mu_r)(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{(u\circ\mu_r)(e^{i\theta})}_{u(re^{i\theta})} \cdot P_a(\theta-\psi)\, d\theta$

Using (uniform!) continuity of u on $\overline{D}_1$, we get that as $r \to 1^-$, $u(re^{i\theta}) \to u(e^{i\theta})$ uniformly in $\|\cdot\|_{[0,2\pi]}$ (and thus the same for its product with the continuous function $P_a(\theta-\psi)$, which doesn't vary).

Taking limits on both sides now gives the desired result. $\square$

Corollary Under the above hypotheses, the harmonic conjugate function of u is given (up to a constant) by
$$v(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, \mathrm{Im}\left\{\frac{e^{i\theta}+\alpha}{e^{i\theta}-\alpha}\right\} d\theta .$$

Proof: Define $f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\cdot\frac{e^{i\theta}+\alpha}{e^{i\theta}-\alpha}\,d\theta$

$\left(\text{on } \partial D_1,\ \frac{dz}{z} = d\log z = d(i\theta) = i\,d\theta\right) = \frac{1}{2\pi i}\oint_{\partial D_1} u(z)\cdot\frac{z+\alpha}{z-\alpha}\,\frac{dz}{z}$,

which is† in $\mathcal{H}ol(D_1)$ (as a function of α).

Taking real parts gives the Poisson formula (& hence u);

so taking imaginary parts gives the harmonic conjugate. $\square$

II. The Dirichlet problem for D_1

We can also give "physical" interpretations of the various Poisson formulas: in (1), we can see the "kernel" $\frac{1-|\alpha|^2}{|e^{i\theta}-\alpha|^2}$ (which is itself a harmonic function of α [HW]) as the "point-source-at-$e^{i\theta}$ stable temperature distribution".

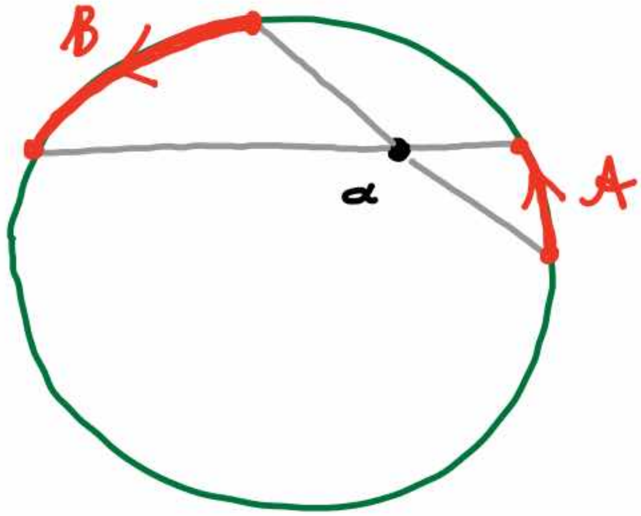
The geometric formula (4) interprets $u(\alpha)$ as the (weighted)

---

† Technically, this uses the following (easy application of Morera):

__Lemma__: Let $\varphi(w,t) \in C^0(U\times[\alpha,\beta])$, with $\varphi(w,t_0) \in \mathcal{H}ol(U)$ $\forall t_0 \in [\alpha,\beta]$. Then $F(w) = \int_\alpha^\beta \varphi(w,t)\,dt \in \mathcal{H}ol(U)$ (i.e. is analytic in w).

average of $u|_{\partial D_1}$ with respect to the measure which weights the arc A by $\frac{1}{2\pi} \times$ (arclength of B) :



In particular, since the constant 1 is a harmonic function, we know that $(\forall \alpha \in D_1)$

$$\text{(**)} \qquad \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|\alpha|^2}{|e^{i\theta}-\alpha|^2} \, d\theta = 1 .$$

We'll now use Poisson's formula to construct harmonic functions.

$\boxed{\text{Theorem}}$ Let $f \in C^0_{\mathbb{R}}(\partial D_1)$, and set

real-valued

$$u(z) := \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1-|z|^2}{|z-e^{i\theta}|^2} \, d\theta & , \ z \in D_1 \\ f(z) & , \ z \in \partial D_1 . \end{cases}$$

Then $u \in C^0(\overline{D}_1)$ and $u|_{D_1} \in \mathcal{H}(D_1)$

The proof proceeds in two steps, for which I'll offer 2 proofs each.

Proof Step 1 [$u|_{D_1}$ is harmonic]:

Quick version: $u(z)$ is Re of $\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta$, which is holomorphic by the lemma in the footnote above.

More explicit version: first note

$$\frac{e^{i\theta}}{e^{i\theta}-z} + \frac{e^{-i\theta}}{e^{-i\theta}-\bar{z}} - 1 = \frac{(1-e^{i\theta}\bar{z}) + (1-ze^{-i\theta}) - (1+|z|^2 - ze^{-i\theta} - \bar{z}e^{i\theta})}{|z-e^{i\theta}|^2}$$

$$= \frac{1-|z|^2}{|z-e^{i\theta}|^2}, \quad \text{so (by Poisson)}$$

$$u(z) = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \frac{e^{i\theta}\,d\theta}{e^{i\theta}-z}}_{\substack{=:\mathrm{I} \\ \text{holomorphic by} \\ \text{representation theorem}}} + \underbrace{\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \frac{e^{-i\theta}\,d\theta}{e^{-i\theta}-\bar{z}}}_{\substack{=:\mathrm{II} \\ \text{antiholo. (conj.} \\ \text{is holo. by representation} \\ \text{theorem)}}} - \underbrace{\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,d\theta}_{\substack{=:\mathrm{III} \\ \text{constant}}}$$

Since u is real,

$$u = \mathrm{Re}\, u = \underbrace{\mathrm{Re}\,\mathrm{I} + \mathrm{Re}\,\mathrm{II} - \mathrm{Re}\,\mathrm{III}}_{\text{all clearly harmonic}}.$$

$\square$

Proof Step 2 [Continuity at the boundary]:

(Intuition) Poisson weighting $\to$ δ-function as z moves radially toward ∂D_1

In particular, $\frac{1-|z|^2}{|e^{i\theta}-z|^2} = 0$ for $\begin{cases} |z|=1 \\ \text{and} \\ z \neq e^{i\theta} \end{cases}$.

Slick proof: For a piecewise C^0 function $\tilde{f}$ on ∂D_1,

set $P_{\tilde{f}}(z) := \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(e^{i\theta}) \frac{1-|z|^2}{|e^{i\theta}-z|^2} d\theta$.

By the Lemma, if we write $\gamma := \overline{\{z \in \partial D_1 \mid \tilde{f}(z) \neq 0\}}$

then $P_{\tilde{f}} \in \mathcal{H}(\mathbb{C}\setminus\gamma)$. Fix $e^{i\theta_0} \in \partial D_1$, and assume $f(e^{i\theta_0}) = 0$. (It suffices to consider this case, as we can add a constant to u later.) Choose $\gamma_2 :=$ small open arc of ∂D_1 such that $\gamma_2 \ni e^{i\theta_0}$ and $\|f\|_{\gamma_2} < \frac{\epsilon}{2}$;

set $\gamma_1 := \partial D_1 \setminus \gamma_2$. Put $\tilde{f}_i := \begin{cases} f & \text{on } \gamma_i \\ 0 & \text{on } \gamma_{3-i} \end{cases}$. Then

$$\left| P_{\tilde{f}_2}(z) \right| \leq \frac{\epsilon}{2} \cdot \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2} d\theta \underset{(***)}{=} \frac{\epsilon}{2} \text{ for } z \in D_1.$$

Now $\exists\, \delta > 0$ s.t. (for $z \in D_1$)

i.e. $e^{i\theta} \in \gamma_1$

$$|z - e^{i\theta_0}| < \delta \implies \left\| f(e^{i\theta}) \frac{1-|z|^2}{|z-e^{i\theta}|^2} \right\|_{\gamma_1} < \epsilon/2$$

$$\Longrightarrow \left| P_{\tilde{\gamma}_1}(z) \right| < \epsilon/2 .$$

So $u = P_{\tilde{\gamma}_1} + P_{\tilde{\gamma}_2}$ (on D_1) $\Longrightarrow$

$|u(z)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for $z \in D(e^{i\theta_0}, \delta) \cap D_1$,

which (since $u(e^{i\theta_0}) = f(e^{i\theta_0}) = 0$) is the desired continuity statement.

More explicit approach: Fix θ_0, write $z = re^{i\theta} \in D_1$.

Then

$$u(e^{i\theta_0}) - u(z) = \frac{1}{2\pi}\int_0^{2\pi} \left\{ f(e^{i\theta_0}) - f(e^{i\psi}) \right\} \frac{1-r^2}{|1-re^{i(\theta-\psi)}|^2} d\psi$$

(use $\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\psi)\, d\theta = 1$
& definition of u)

Put $M = \|f\|_{\partial D_1}$, and pick $\epsilon > 0$, $(1>)\delta > 0$ s.t.

$$|\eta_1 - \eta_2| < \delta \Longrightarrow |f(e^{i\eta_1}) - f(e^{i\eta_2})| < \epsilon/2$$

(can do this by [uniform] continuity of f on ∂D_1);

also pick r, θ s.t. $|\theta - \theta_0| < \delta/3$, and

$\max\left(\frac{1}{2}, 1 - \frac{\delta^2 \epsilon}{100M}\right) < r < 1$. (That is, we are taking $z = re^{i\theta} \in D_1$ close to $e^{i\theta_0}$.) Then

$$|u(e^{i\theta_0}) - u(z)| \leq \frac{1}{2\pi}\int_{|\psi-\theta_0|<\delta} |f(e^{i\theta_0}) - f(e^{i\psi})| \frac{1-r^2}{|1-re^{i(\theta-\psi)}|^2} d\psi$$

$$+ \frac{1}{2\pi}\int_{|\psi-\theta_0|\geq\delta} \quad " \quad " \quad " \quad "$$

$$\left.\begin{aligned}&|\theta-\theta_0| < \frac{\delta}{3}\\ &\text{and}\\ &|\psi-\theta_0| \geq \delta\end{aligned}\right\} \Rightarrow |\theta-\psi| \geq \frac{2\delta}{3}.$$

If $\mu \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ & $|\mu| \geq \frac{2\delta}{3}$ then

$$\begin{aligned}|1-re^{i\mu}|^2 &= (1-re^{i\mu})(1-re^{-i\mu})\\ &= 1+r^2-2r\cos\mu\\ &= (1-r)^2+2r(1-\cos\mu)\\ &\geq 2r\left(1-\left[1-\frac{\mu^2}{2!}+\frac{\mu^4}{4!}\right]\right)\\ &= r^2\mu\left(1-\frac{\mu^2}{12}\right) \geq \frac{r\mu^2}{2} \geq \frac{\mu^2}{4}\\ &\geq \frac{\delta^2}{9}\end{aligned}$$

, and obviously this inequality holds for larger $|\mu| \in [\pi/2, \pi]$.

$$\leq \frac{1}{2\pi}\int_{|\psi-\theta_0|<\delta} \frac{\epsilon}{2}\cdot\frac{1-r^2}{|1-re^{i(\theta-\psi)}|^2}\,d\psi$$

$$+\frac{1}{2\pi}\int_0^{2\pi} 2M\cdot\frac{(1+r)(1-r)}{\delta^2/9}\,d\psi$$

$1+r < 2$

$1-r < \dfrac{\delta^2\epsilon}{100M}$

$$\leq \frac{\epsilon}{2} + \underbrace{\frac{18\cancel{M}}{\cancel{2\pi}\cdot\cancel{\delta^2}}\cdot\frac{2\cancel{\delta^2}\epsilon}{100\cancel{M}}\cdot\cancel{2\pi}}_{\frac{36}{100}\epsilon}$$

$$< \epsilon.$$

$\square$