Lecture 30 : Harmonic functions

I. Motivation

Harmonic functions crop up in thermodynamics, electrostatics, fluid mechanics, and many other "natural" settings.

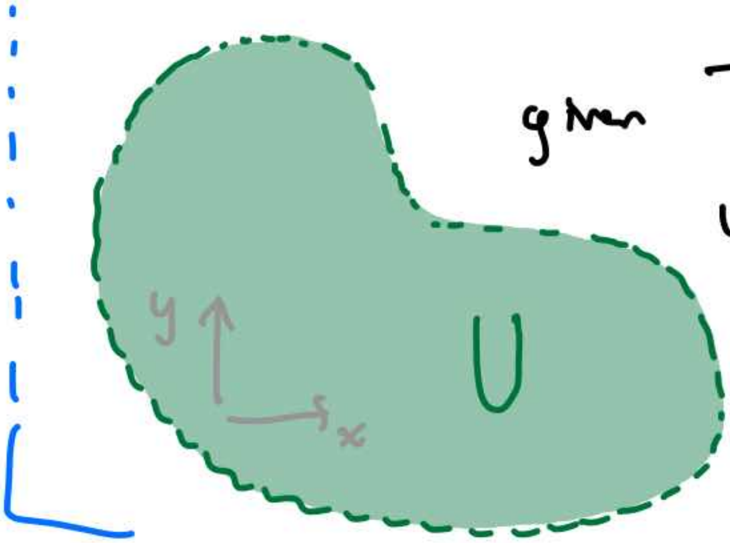
Example // The heat equation describes how the rate of change of temperature T at a point depends on nearby temperature,

$$\frac{\partial T}{\partial t} = k \cdot \Delta T ,$$

where (in 2-D) $T = T(t, x, y)$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ (Laplacian), k = thermal diffusivity of the material. Hence,

$$\Delta T \equiv 0 \implies T \text{ stable in time.}$$

Obviously, a constant solves this equation; but physical intuition says that if you place a heat source (or cold source [e.g., Children]) at the boundary of a region V, after a long time elapses T should stabilize, yielding a solution of $\Delta T \equiv 0$.



given $T_0 : \partial U \to \mathbb{R}$ (heat source),
want $T : \overline{U} \to \mathbb{R}$ C^0,
with $\begin{cases} T\big|_{\partial U} \equiv T_0, \\ \Delta T \equiv 0 \text{ on } U. \end{cases}$ //

More generally, let $U \subset \mathbb{C}$ be an open set.

$\boxed{\text{Definition}}$ $u : U \to \mathbb{R}$ is harmonic $\iff$

$$u \in C^2(U),\ \Delta u \equiv 0 .$$

We write $u \in \mathcal{H}(U)$. Note that this is a real-valued function of a complex variable (or 2 real ones).

$\boxed{\text{Dirichlet "boundary-value" problem}}$ Given $u_0 : \partial U \to \mathbb{R}$ piecewise C^0, when does there exist a unique (piecewise C^0) $u : \overline{U} \to \mathbb{R}$ harmonic on U and equal to u_0 on ∂U?

(NOT always!)

- only reasonable to ask for this where u_0 is continuous (or at least, "interpolates the discontinuity" as in Fourier series: )

- fails too for some cases of non-simply-connected U and continuous u_0, e.g. D_1^*
- always soluble for, say, u continuous & U simply connected.

Example // <u>Electrostatics (voltage)</u>: place an electric charge on ∂U; the induced electric potential function "solves Dirichlet". Proof by nature! If you add electric charges au_0 & bv_0 on the boundary, then the solutions add — $au+bv$. (Principle of superposition.) This illustrates that linear combinations of harmonic functions are harmonic. Products are <u>NOT</u>, e.g. $x \in \mathcal{H}(\mathbb{C})$ but $x^2 \notin \mathcal{H}(\mathbb{C})$. //

Our first step below is to prove a relation between harmonic & holomorphic functions that will bring to bear all of the techniques of complex analysis.

II. Harmonic conjugates

Let $f \in \mathrm{Hol}(U)$, $f = u + iv$. (Of course, u & v are C^∞.)

By the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Big\downarrow \partial/\partial x \qquad \Big\downarrow \partial/\partial y$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} v = \frac{\partial}{\partial y}\frac{\partial}{\partial x} v = -\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u \in \mathcal{H}(U).$$

$\boxed{\text{Proposition}}$ The real (& imaginary) part of a holomorphic function is harmonic.

Examples//

$$\left.\begin{array}{l} ax + by + c \\ e^x \cos y \\ x^2 - y^2 \end{array}\right\} \text{ on } \mathbb{C}$$

$$\left.\begin{array}{l} \frac{1}{2}\log(x^2+y^2) \\ \frac{x}{x^2+y^2} \end{array}\right\} \text{ on } \mathbb{C}^*$$

//

Conversely, let U be simply connected and $u \in \mathcal{H}(U)$.

[Recall that: • $\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

• f holo. $\Leftrightarrow$ $\partial f/\partial \bar{z} = 0$ $\Leftrightarrow$ $\partial f/\partial z = \frac{df}{dz} (=: f')$.]

We have $\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \frac{1}{4}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) = \frac{1}{4}\Delta$.

Set $g := 2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$, so that

$$\frac{\partial g}{\partial \bar{z}} = 2\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}u = \frac{1}{2}\Delta u \underset{(u \in \mathcal{H}(U))}{=} 0$$

$$\Rightarrow g \in Hol(U) \underset{U \text{ simp. conn.}}{\Longrightarrow} g = f', \quad f \in Hol(U).$$

Write $f = u_1 + iv_1$. Then

$$g = f' = \frac{\partial f}{\partial z} = \frac{1}{2}\Big(\underbrace{\frac{\partial}{\partial x}(u_1 + iv_1)}_{\frac{\partial u_1}{\partial x}} - \underbrace{i\frac{\partial}{\partial y}(u_1 + iv_1)}_{+\frac{\partial v_1}{\partial y} = \frac{\partial u_1}{\partial x}}\Big)$$

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = g \qquad = \frac{\partial u_1}{\partial x} + i\left(\frac{-\partial u_1}{\partial y}\right)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u_1}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u_1}{\partial y} \Longrightarrow u_1 = u + C \text{ (real const.)}$$

Set $F = f - C$; then $Re(F) \equiv u$ on U.

<u>Uniqueness</u>: If $F_0 \in Hol(U)$, $Re(F_0) \equiv u$,

then $\mathrm{Re}(F-F_0) \equiv 0$ on U, i.e. $F-F_0$ is a holo. fcn. mapping to a straight line. This contradicts OMT unless $F-F_0$ is an (imaginary) constant.

$\boxed{\text{Prop. / Defn.}}$ A harmonic function u on a simply connected open set is the real part of a holomorphic function there (unique up to an imaginary constant), whose imaginary part is called the harmonic conjugate of u. (Its level curves are perpendicular to those of u.)

Example // (harmonic conjugates of last example, up to const.)

$-bx + ay$

$e^x \sin y$

$2xy$

$\arctan(y/x)$ ← not well-defined on $\mathbb{C}^*$, although $\log(x^2+y^2)$ was. This illustrates the necessity of the hypothesis of simple connectivity. //

$\dfrac{-y}{x^2+y^2}$

Etymology: Why "harmonic"? Consider $u \in \mathcal{H}(D_{1+\epsilon})$, and restrict to C_1:

$$u = \mathrm{Re}\Big(\sum \underbrace{\alpha_n}_{a_n + i b_n} z^n\Big)$$

$$u(e^{i\theta}) = \mathrm{Re}\left(\sum_n (a_n + ib_n)\left(\cos(n\theta) + i\sin(n\theta)\right)\right)$$

$$= \sum_n \left(a_n \cos(n\theta) - b_n \sin(n\theta) \right)$$

"circular harmonics" — think of harmonic overtones on a vibrating string.

This also suggests a relation between Fourier series and the Dirichlet problem for D_1, which I'll make more precise when we discuss the latter.

III. Basic properties

We start by picking some <u>really</u> low-hanging fruit.

Corollary 1 Harmonic functions are C^∞.

<u>Pf</u>: $u = \mathrm{Re}(f)$, f holo. $\Rightarrow f' = \frac{\partial f}{\partial x} \underset{CR}{=} \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ holo.

$\Rightarrow \partial u/\partial x,\ \partial u/\partial y$ harmonic; start over. $\square$

Corollary 2 Given $g: U \to V$ holo. (or antiholo.) & $u: V \to \mathbb{R}$ harmonic, $u \circ g : U \to \mathbb{R}$ is harmonic.

<u>Pf</u>: $u = \mathrm{Re}(f)$, f holo. $\Rightarrow f \circ g$ holo. [or antiholo.*]

$\Rightarrow \mathrm{Re}(f \circ g) = u \circ g \in \mathcal{H}(U)$. $\square$

$*\ \frac{\partial}{\partial z}(f \circ g) = \frac{\partial f}{\partial z}\cancel{\frac{\partial g}{\partial z}}^{\,0} + \cancel{\frac{\partial f}{\partial \bar z}}\frac{\partial \bar g}{\partial z} = 0 \Rightarrow f \circ g$ antiholo.

(f holo., g antiholo.)

(MAXIMUM PRINCIPLE)

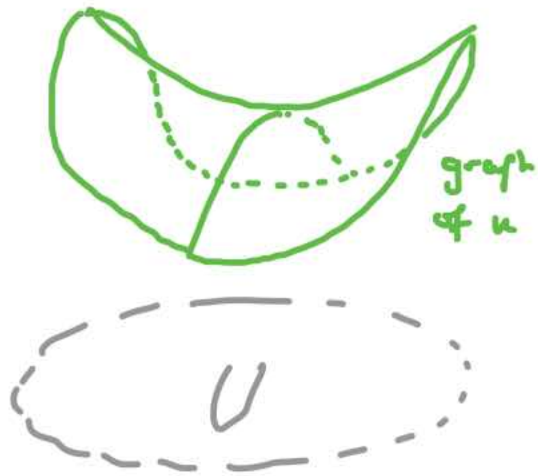
$\boxed{\text{Corollary 3}}$ U open, connected; $u \in C^0(\overline{U})$, $u|_U \in \mathcal{H}(U)$.

If $u(z_0) \geq u(z)$ $\forall z \in \overline{U}$, then $z_0 \in \partial U$ or u is constant.

(or $\leq$: minimum principle)

Pf: If U is simply connected, $\exists f \in \mathrm{Hol}(U)$, $\mathrm{Re}(f) = u$

$\Rightarrow |e^f| = |e^u e^{iv}| = e^u$. Apply MMP. More generally, U is a union of simply connected open sets and the result follows easily from the simply connected case. $\square$



So the values taken by u on U lie strictly between $\max_{z \in \partial U}(u(z))$ and $\min_{z \in \partial U}(u(z))$ — resembling a soap film (not entirely correct, since graphs of harmonic functions are not minimal surfaces).

Recall that $\Delta u \equiv 0$ means "u is a time-stable temperature function". So it should seem reasonable that at any point, its value is the <u>average</u> of nearby values.

$\boxed{\text{MVT}}$ $u \in \mathcal{H}(U)$, $U \supset \overline{D}(z_0, r)$ $\Rightarrow$

(Mean Value Thm.)

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

Proof: $\exists f \in \mathcal{H}ol(D(z_0, r+\epsilon))$ with $\mathrm{Re}(f) = u$.

By Cauchy $\int$ formula,

$$\left(\begin{aligned} w &= z_0 + re^{i\theta} \\ dw &= ire^{i\theta}d\theta \end{aligned}\right)$$

$$\underbrace{u(z_0)}_{\rotatebox{90}{=}} + iv(z_0) = f(z_0) = \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(w)}{w - z_0}\,dw = \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{\cancel{re^{i\theta}}}\,\cancel{i}\cancel{re^{i\theta}}\,d\theta$$

$$= \underbrace{\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta}_{} + i\left(\cdots\right). \qquad \square$$

In studying holomorphic functions we used Cauchy to prove Liouville's theorem. To prove a version for harmonic functions, one could try to show $\underbrace{u \text{ bounded} \in \mathcal{H}(\mathbb{C}) \Rightarrow \text{harmonic conjugate also bounded}}$. But in fact we get a stronger result from arguing as follows.

$\boxed{\text{Corollary 4}}$ $u \in \mathcal{H}(\mathbb{C})$, bounded above OR below $\Rightarrow$ u constant.

Proof 1: $\exists f \in \mathcal{H}ol(\mathbb{C})$, $\mathrm{Re}(f) = u \Rightarrow e^{\pm f}$ holo., $|e^{\pm f}| = e^{\pm u}$ bounded!

Use Liouville. $\square$

Proof 2 (direct): Say $u \geq 0$; $u(\overset{\in \mathbb{C}}{z_0}) - u(0) = \frac{1}{\pi r^2}\left(\iint_{D(z_0,r)} u\,dA - \iint_{D(0,r)} u\,dA\right)$

by MVT (and using $dA = r\,dr\,d\theta$). So

$$|u(z_0) - u(0)| \leq \frac{1}{\pi r^2}\iint u\,dA \overset{\text{MVT}}{\leq} \frac{1}{\pi r^2}\left\{-\pi(r - |z_0|)^2 u(0) + \pi(r + |z_0|)^2 u(0)\right\}$$

the nonoverlapping parts of $D(z_0,r)$ & $D(0,r)$ are contained in $\xrightarrow{r - |z_0| < |z| < r + |z_0|}$

$$= u(0)\cdot\frac{4r|z_0|}{r^2} \to 0 \ (r \to \infty). \qquad \square$$