Lecture 3: More on Cauchy-Riemann

I. $\partial/\partial z$ and $\partial/\partial \bar{z}$

We begin with an alternate form of the C-R equations: set

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Think of these as operating on some

$$f : \underset{\mathbb{C}}{\underset{\cong}{U}} \longrightarrow \mathbb{C}$$

which may or may not be complex differentiable; however we do assume that the partials $\partial/\partial x$ & $\partial/\partial y$ of $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ exist and are continuous. (That is, "f is continuously $\mathbb{R}$-differentiable.")

<u>If</u> f is holomorphic, then

$$\frac{\partial}{\partial z} f = \frac{1}{2}\left\{ \frac{\partial}{\partial x}(u+iv) - i\frac{\partial}{\partial y}(u+iv) \right\}$$

$$\underset{\text{C-R}}{=} \frac{1}{2}\left\{ \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right\}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(u+iv)$$

$$= \frac{df}{dz},$$

by "independence of direction" of $h \to 0$

the $\mathbb{C}$-derivative.

By similar means, one has (if f is holomorphic)

$$\frac{\partial}{\partial \bar z} f = \frac{1}{2}\left\{ \frac{\partial}{\partial x}(u+iv) + i\frac{\partial}{\partial y}(u+iv) \right\} = 0.$$

So we have (you should check details):

$$\boxed{f \text{ holomorphic} \iff \frac{\partial f}{\partial z} = \frac{df}{dz} \iff \frac{\partial f}{\partial \bar z} = 0.}$$

Regardless of holomorphicity of f, $\partial/\partial z$ and $\partial/\partial \bar z$ obey the usual sum, product, and quotient rule (as $\partial/\partial x$ and $\partial/\partial y$ do).

What about the chain rule? Assume $f, g : U \to \mathbb{C}$ are continuously $\mathbb{R}$-differentiable; then by writing out $\partial/\partial z$, $\partial/\partial \bar{z}$ in terms of $\partial/\partial x$ & $\partial/\partial y$ (& using the multivariable chain rule) one finds that

$$\frac{\partial}{\partial z}(f \circ g)(z) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial z}(z) + \frac{\partial f}{\partial \bar{z}}(g(z))\frac{\partial \bar{g}}{\partial z}(z)$$

(where one can also change the 3 circled "z"'s simultaneously to "$\bar{z}$"). A corollary is that if f or g is holomorphic, then

$$\frac{\partial}{\partial z}(f \circ g)(z) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial z}(z).$$

Recovering the usual partials: $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$, $\frac{\partial}{\partial y} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \bar{z}}$.

If U is a connected open set, f is holomorphic, and $\underbrace{f' \equiv 0}_{\text{identically zero}}$, then f is constant.

Proof (idea): Any 2 points of U are connected by a chain of horizontal & vertical segments, and

- f holomorphic $\Rightarrow \frac{\partial}{\partial \bar{z}} f \equiv 0$
- $f' \equiv 0 \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \underbrace{\cancel{\frac{\partial f}{\partial \bar{z}}}}_{0} = f' \equiv 0$
- similarly, $\frac{\partial f}{\partial y} \equiv 0$.

$\square$

Example 1 $f(z) = e^z = \overbrace{e^x\cos y}^{u} + i\overbrace{e^x \sin y}^{v}$

is holomorphic: $\partial u/\partial x = e^x \cos y = \partial v/\partial y$

$\partial v/\partial x = e^x \sin y = -\partial u/\partial y$.

Example 2 $f(z) = \bar{z} = x - iy$ (so $u = x$, $v = -y$)

is not holomorphic: $\partial u/\partial x = 1 \neq -1 = \partial v/\partial y$.

Moreover, $\frac{\partial}{\partial \bar{z}}\bar{z} = 1$. (Other non-holo. functions include $|z|$, $|z|^2$.)

Example 3 In contrast,

$$\frac{\partial}{\partial \bar{z}} z = \frac{1}{2}\left\{\frac{\partial z}{\partial x} + i\frac{\partial z}{\partial y}\right\} = \frac{1}{2}\left\{1 + i(i)\right\} = 0 \Rightarrow z \text{ holo.}$$

Since $\partial/\partial\bar{z}$ obeys "$(\alpha f + \beta g)' = \alpha f' + \beta g'$", Leibniz, & quotient rules, $\partial/\partial\bar{z}$ of every polynomial or (more generally) rational function of z is ZERO, making such functions holomorphic (where defined).

Further, Leibniz's rule + induction + $\frac{\partial z}{\partial z} = 1$

$$\Rightarrow (z^n)' = \frac{\partial}{\partial z} z^n = n z^{n-1}.$$

II. Rational functions †

Let $P(z)$, $Q(z)$ be polynomials with no common factors, and put

$$R(z) := P(z)/Q(z) \quad ;$$

we may extend this to a map

$$R : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$$

by setting

$$R(\text{roots of } Q) := \infty$$

$$R(\infty) := \lim_{w\to 0} R\left(\tfrac{1}{w}\right).$$

also a rational function

Define

- $\mathrm{ord}_{z_0}(R) :=$ the unique integer k making $(z-z_0)^{-k} R(z)$ finite & nonzero at $z = z_0$
- $\mathrm{ord}_{\infty}(R) :=$ the unique integer k making $w^{-k} R\left(\tfrac{1}{w}\right)$ finite & nonzero at $w = 0$

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† In this section, we assume the Fundamental Theorem of Algebra, which will be proved later.

- $\deg(R) := \max\{\deg P, \deg Q\}$

(Ahlfors: "order")

degree as polynomials

$$= \underbrace{\sum_{\substack{z\in\hat{\mathbb{C}} \\ \mathrm{ord}_z(R)>0}}}_{\text{zeroes}} \mathrm{ord}_z(R)$$

$$= -\underbrace{\sum_{\substack{z\in\hat{\mathbb{C}} \\ \mathrm{ord}_z(R)<0}}}_{\text{poles}} \mathrm{ord}_z(R)$$

e.g., if $\deg P > \deg Q$ then R has a pole of order $\deg P - \deg Q$ at ∞.

$=$ # of solutions (with multiplicity) of $R(z) = \alpha$ for any $\alpha \in \mathbb{C}$

$R(z)-\alpha$ has same poles hence degree as $R(z)$

Let z_j be the finite poles of $R(z)$.

Theorem (partial fractions decomposition)

There exist polynomials P_j (of degree $(-\mathrm{ord}_{z_j}(R))$)

such that $R(z) = \sum_j P_j\left(\frac{1}{z-z_j}\right) + P_\infty(z)$.

(One has the term $P_\infty(z) \iff \mathrm{ord}_\infty(R) < 0$.)

Proof: The rational function $R(z_j + \frac{1}{w})$ has a pole at $w = \infty$; hence $\deg(\text{num}) > \deg(\text{denom.})$ and long division yields

$$R(z_j + \tfrac{1}{w}) = P_j(w) + \underbrace{R_j(w)}_{\substack{\text{with } \deg(\text{num}) \le \deg(\text{denom.}) \\ \Rightarrow \text{ finite at } \infty}}$$

$\Longrightarrow$ $\left(\begin{array}{l} z = z_j + \frac{1}{w} \\ w = \frac{1}{z - z_j} \end{array}\right)$ $R(z) - P_j\left(\frac{1}{z - z_j}\right)$ is finite at z_j, and no other poles have been introduced (or changed in order).

Continue process for each remaining z_j's, so that finally

$$R(z) - \sum_j P_j\left(\frac{1}{z - z_j}\right)$$

has <u>no</u> poles in $\mathbb{C}$, and so must be a polynomial. $\square$

To conclude, we take a look at polynomials

$$P(z) = \alpha_n \prod_{j=1}^{n} (z - z_j) = \sum_{k=0}^{n} \alpha_k z^k .$$

Writing

$$P'(z) = \beta_{n-1} \prod_{\ell=1}^{n-1} (z - w_\ell) = \sum_{k=1}^{n} \underbrace{k\alpha_k}_{\beta_{k-1}} z^{k-1},$$

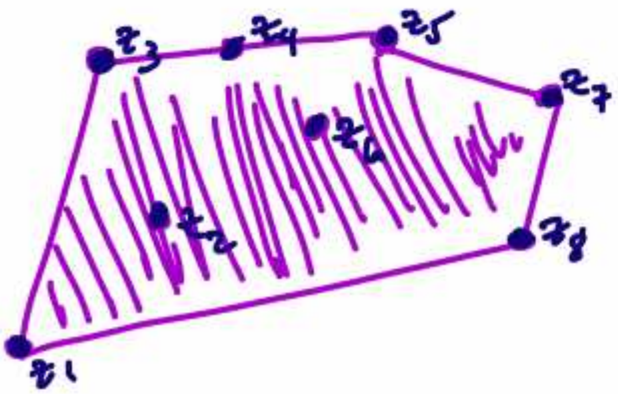
we notice that the average† of the zeroes in $\mathbb{C}$

$$\frac{\sum w_\ell}{n-1} = \frac{-\beta_{n-2}}{(n-1)\beta_{n-1}} = \frac{-\cancel{(n-1)}\,\alpha_{n-1}}{\cancel{(n-1)}\, n\, \alpha_n} = \frac{\alpha_{n-1}}{n\alpha_n} = \frac{\sum z_j}{n}$$

doesn't change with differentiation of P.

This suggests looking at the <u>convex hull</u> of the zeroes, i.e. the smallest convex polytope $\subseteq \mathbb{C}$ containing all the z_j's.

e.g.



$\Delta(P)$
(convex hull)

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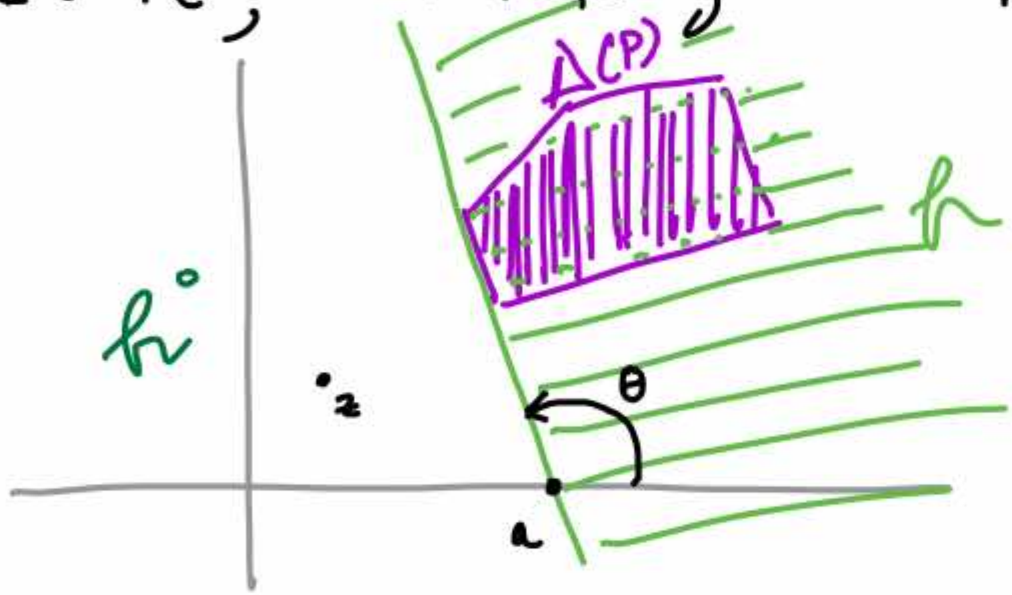
† This is weighted by multiplicity (by construction), so is a "center of mass" of the zeroes (if you will)

$\boxed{\text{Lucas's Theorem:}}$ $\underline{\Delta(P') \subseteq \Delta(P)}$

Proof: $\Delta(P)$ is an intersection of half-planes; so it suffices to show each w_j belongs to any half-plane containing all the $\{z_j\}$. Put differently, given any closed half-plane h containing the $\{z_j\}$, we must show that <u>P' has no zeroes in the (open) complement h°</u>. Since P has none there, we can check the assertion for a constant times

$$\frac{P'}{P} = \sum_{j=1}^{n} \frac{1}{z - z_j}.$$

If $z \in h^\circ$, then referring to the picture



$$0 < \mathrm{Im}\left(\frac{z-a}{e^{i\theta}}\right) = \underbrace{\mathrm{Im}\left(\frac{z_j-a}{e^{i\theta}}\right)}_{\leq 0 \text{ since } z_j \in h} + \mathrm{Im}\left(\frac{z-z_j}{e^{i\theta}}\right)$$

$$\leq \mathrm{Im}\left(\frac{z-z_j}{e^{i\theta}}\right)$$

$$\Rightarrow 0 > \mathrm{Im}\left(\frac{e^{i\theta}}{z-z_j}\right)$$

imaginary part of reciprocal has opposite sign: $be^{i\phi} \mapsto \frac{1}{b}e^{-i\phi}$

Thus

$$\mathrm{Im}\left(\frac{e^{i\theta}p'}{p}\right) = \sum_{j=1}^{n} \mathrm{Im}\left(\frac{e^{i\theta}}{z-z_j}\right) < 0,$$

which completes the proof. $\square$