Lecture 29 : Integral transforms

We are interested in these mainly as a source of integrals on $\mathbb{R}$ for which residue theory is useful. In general, a "transform" is just a $\mathbb{C}$-linear transformation of $\mathbb{C}$-valued functions on the real line :

$$(Tf)(k) := \int f(t)\, G(k,t)\, dt .$$

I. Three famous types

(i) Fourier : $\hat{f}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$

(ii) Laplace : $(\mathcal{L}f)(k) := \int_0^\infty f(x)\, e^{-kx}\, dx$

(iii) Mellin : $(\mathcal{M}f)(k) := \int_0^\infty f(x)\, x^{k-1}\, dx$.

Of course, while a priori k is a real variable, sometimes we will want to make these transforms functions on the entire complex plane.

Example// (The Gamma function)

Let $f(x) = e^{-x}$. Then we define

$$\Gamma(k) := (\mathcal{M}f)(k) = \int_0^\infty e^{-x} x^{k-1}\,dx$$

$$= e^{-x}\frac{x^k}{k}\Big|_0^\infty - \int_0^\infty \frac{x^k}{k}(-e^{-x})\,dx$$

$\int$ by parts :
$u = e^{-x} \rightarrow du = -e^{-x}dx$
$dv = x^{k-1}dx \rightarrow v = x^k/k$

$$= \frac{1}{k}\Gamma(k+1),$$ and note that

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1$$
$$\Gamma(2) = 1\cdot\Gamma(1) = 1$$
$$\Gamma(3) = 2\cdot\Gamma(2) = 2 = 2!$$
$$\Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2 = 3!$$

$$\Rightarrow \Gamma(k) = (k-1)! \quad (\forall\, k \in \mathbb{N}).$$

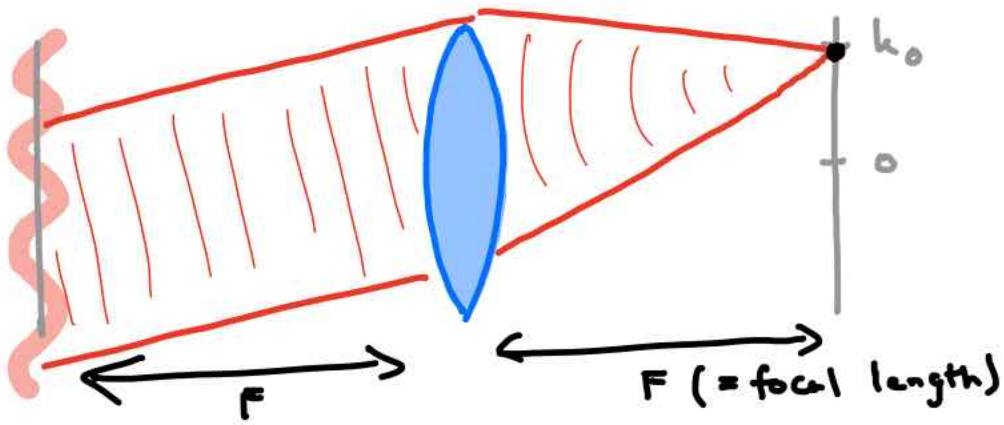
More generally the integral converges (and Γ is holo.) for $\mathrm{Re}(k) > 0$, and we can analytically continue to get a function in $\mathrm{Mer}(\mathbb{C})$. //

Remark: The analytic continuation to $Mer(\mathbb{C})$ uses a "functional equation" $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$. More on this later in the course. //

II. Fourier transforms

First, a bit of intuition.

Recall that $\hat{f}(k)$ quantifies the "infinitesimal amplitude" of a frequency k in $f(x)$. One can think about this in terms of lenses



where a pure frequency

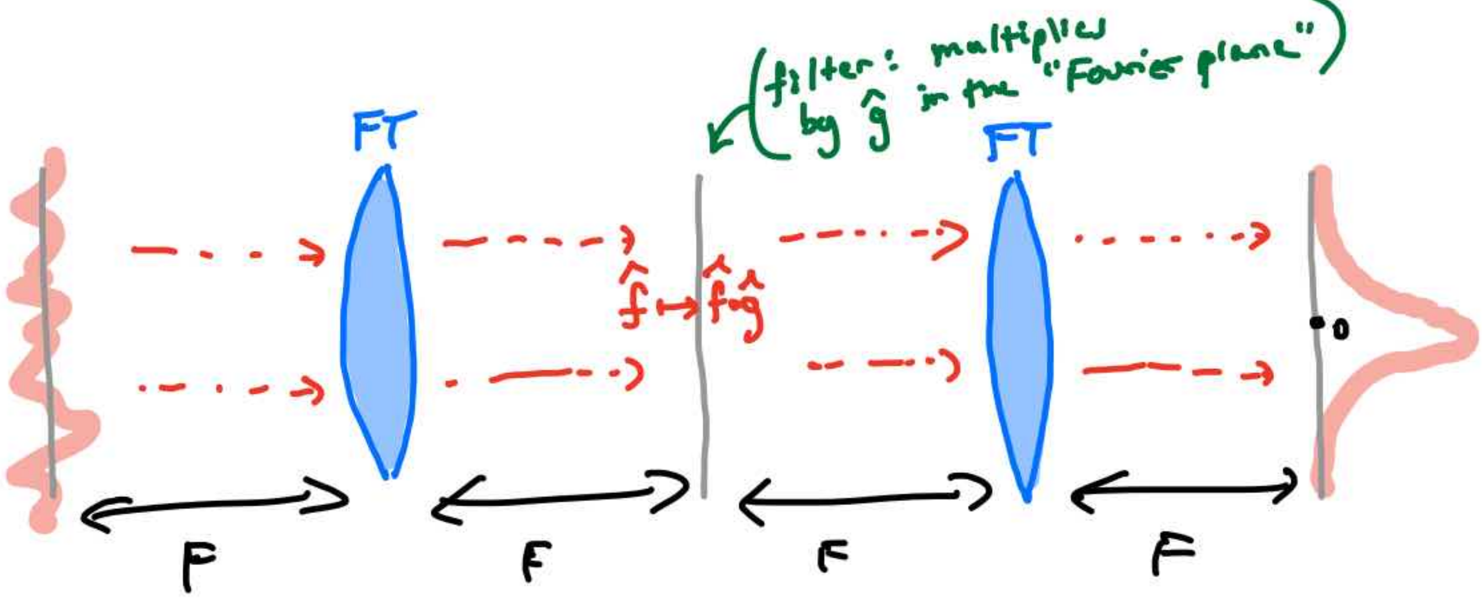
$$e^{ik_0 x} \xrightarrow{FT} \int_{-\infty}^{\infty} e^{i(k_0-k)x}\,dx = \text{"}\delta(k-k_0)\text{"}$$

corresponds on the left to a tilted wavefront.

If you FT a function and multiply with a second function's FT, you get upon FT'ing the result the convolution

$$\widehat{\hat{f}\cdot\hat{g}} = (f*g)(y) := \int_{-\infty}^{\infty} f(x)g(y-x)\,dx.$$

Roughly speaking, the closer f & g are to being the same function, the more $f*g$ is like a spike at $y=0$. In optics, this is used for image detection:

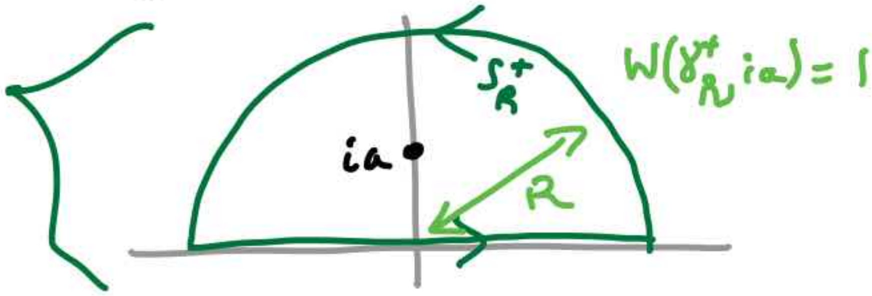


Also, from the "frequency" intuition: a more "concentrated" function should have higher frequencies so have more "spread-out" FT. We'll test this hypothesis below.

Example

$$I(k) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{a^2+x^2}\,dx, \quad \begin{cases} k \in \mathbb{R} \\ a > 0. \end{cases}$$

For $k \geq 0$ use γ_R^+ and the bound



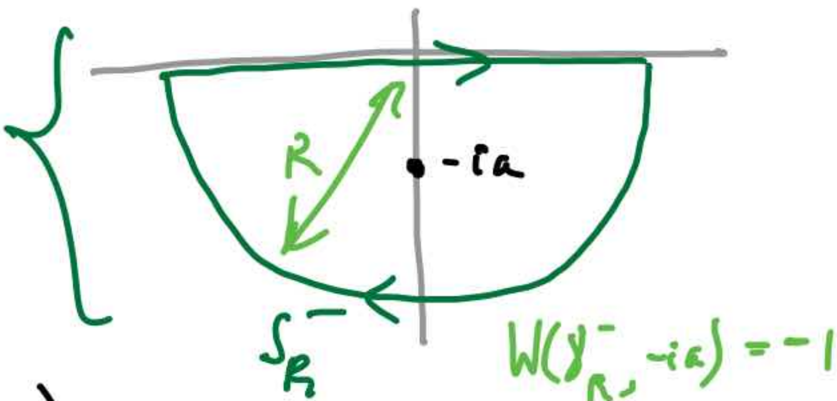
$$\left| \int_{S_R^+} \frac{e^{ikz}}{a^2+z^2}\,dz \right| \leq \frac{\pi R}{R^2-a^2} \longrightarrow 0 \quad (\text{as } R \to \infty)$$

to conclude that

$$I(k) = \lim_{R\to\infty} \int_{\gamma_R^+} \frac{e^{ikz}}{a^2+z^2}\,dz = 2\pi i \operatorname{Res}_{ai}\left(\frac{e^{ikz}/(z+ai)}{z-ai} \right)$$

$$= 2\pi i \cdot \frac{e^{ik(ai)}}{ai+ai} = \frac{\pi}{a} e^{-ak} = \frac{\pi}{a} e^{-|k|a}.$$

For $k \leq 0$ use γ_R^- to get



$$I(k) = -2\pi i \operatorname{Res}_{-ai}\left(\frac{e^{ikz}/(z-ai)}{z+ai} \right)$$

$$= -2\pi i \frac{e^{ik(-ai)}}{-2ai} = \frac{\pi e^{ka}}{a} = \frac{\pi}{a} e^{-|k|a}.$$

So if $f(x) = \dfrac{1}{a^2+x^2}$, $\boxed{\hat{f}(k) = \frac{1}{\sqrt{2\pi}} I(-k) = \sqrt{\frac{\pi}{2}}\, \frac{e^{-|k|a}}{a}.}$ $(\forall k)$

More interestingly, set $F(x) = \frac{a}{a^2+x^2}$, so $\hat{F}(k) = \sqrt{\frac{\pi}{2}}\, e^{-|k|a}$. The area underneath $F(x)$ is independent of a, but "small a" $\Rightarrow$ "F spikier", so more high-frequency stuff. Therefore we expect $\hat{F}$ more spread out, which is indeed the case. //

Here is a more general result on FTs: writing $z = x+iy$, we have the

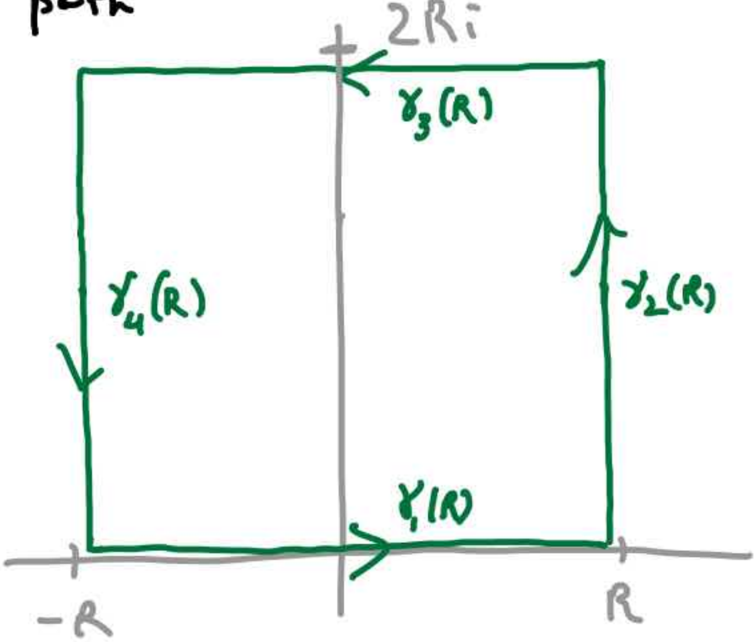
$\boxed{\text{Theorem}}$ Assume $f(z) \in \mathrm{Mer}(\mathbb{C})$ with only finitely many poles, none of these on $\mathbb{R}$. Then

$$\int_{-\infty}^{\infty} f(x) e^{ikx}\,dx = 2\pi i \times \begin{cases} \sum_{y>0} \mathrm{Res}_z\left(f(z)e^{ikz}\right), & k>0 \\ \\ \sum_{y<0} \mathrm{Res}_z\left(f(z)e^{ikz}\right), & k<0. \end{cases}$$

provided only that $|f(z)| \leq B/|z|$ for $|z| \geq K$.

Remark: Our last example used $|f(z)| \leq B/|z|^2$, like the very first example from Lecture 28. But here we can take advantage of the oscillatory behavior of e^{ikz} for $k \neq 0$ to make the "upper part" of the integral go to zero. This upper part won't, however, be a semicircle! //

Proof: This time we take γ_R^+ to be the rectangular path



Take $k > 0$, $R \gg 0$. Parametrizing $-\gamma_3$ by $\underbrace{t + 2Ri}$ gives

$$\left| \int_{-\gamma_3(R)} e^{ikz} f(z)\,dz \right| = \left| \int_{-R}^{R} e^{ik(t+2Ri)} f(t+2Ri) \cdot \overset{d/dt}{1}\, dt \right|$$

$$\leq \cancel{2R} \cdot e^{-2kR} \cdot \frac{B}{\cancel{2R}}$$

$$= B e^{-2kR} \longrightarrow 0.$$

(as $R \to \infty$)

Next, parametrizing γ_2 by $it+R$, we have

$$\left|\int_{\gamma_2(R)} e^{ikz} f(z)\,dz\right| = \left|\int_0^{2R} \underbrace{e^{ik(it+R)}}_{e^{-kt}e^{ikR}} f(it+R)\, \overset{d/dt}{i}\, dt\right|$$

$$\leq \int_0^{2R} e^{-kt}\frac{B}{R}\,dt$$

$$= \frac{B}{R}\left(1-\frac{e^{-2Rk}}{k}\right) \longrightarrow 0.$$

(as $R\to\infty$)

Same story with $\int_{\gamma_4(R)}$. So

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)e^{ikx}\,dx = \lim_{R\to\infty}\int_{\gamma_R} f(z)e^{ikz}\,dz$$

$$\underset{\text{Res. thm.}}{=} 2\pi i \sum_{y>0} \operatorname{Res}_z\left(f(z)e^{ikz}\right).$$

The $k<0$ case is similar, with γ_R^+ replaced by γ_R^-. $\square$

III. Mellin transforms

Example// Let $f(x) = \frac{1}{1+x^2}$.

We compute $(\mathcal{M}f)(k)$ for $k \in (0,2)$, i.e.

$$I(\alpha) := \int_0^\infty \frac{x^\alpha \, dx}{1+x^2} \quad \text{for } -1 < \alpha < 1.$$

For $\alpha = 0$, $I(0) = \arctan(\infty) - \arctan(0) = \frac{\pi}{2}$.

The method we use now works for $\alpha \in (-1,0) \cup (0,1)$.

For $\alpha \neq 0$, the integrand (regarded as a function of z)

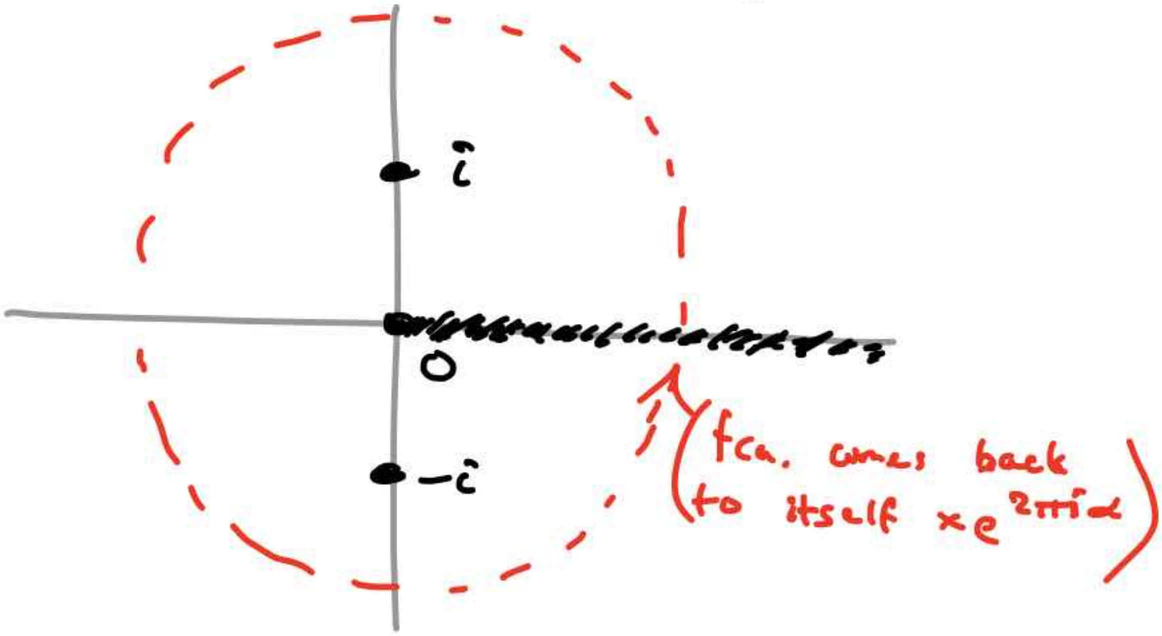
$$F(z) := \frac{z^\alpha}{1+z^2} = \frac{e^{\alpha \log(z)}}{1+z^2}$$

is not single-valued in the complex plane: write $z = re^{i\theta}$, let θ run from 0 to 2π; F doesn't return to itself ——

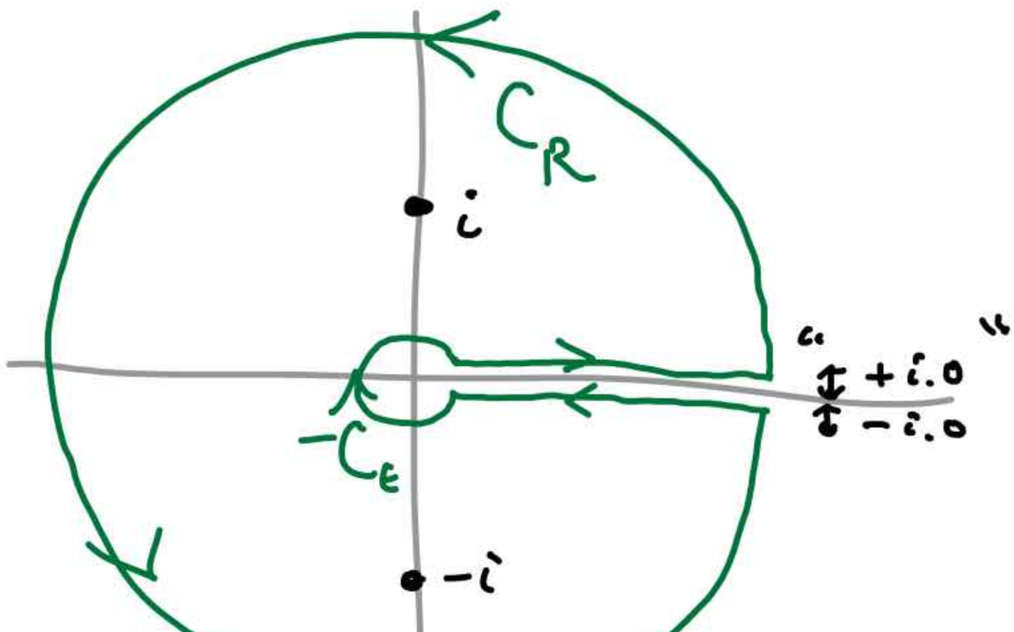
$$F(re^{2\pi i}) \text{ "=" } \frac{r^\alpha}{1+r^2} e^{2\pi i \alpha} = F(r\cancel{e^{0i}}) \cdot \underbrace{e^{2\pi i \alpha}}_{\neq 1}.$$

Regard F as a single-valued function on the cut plane:

$$F \in \mathcal{H}ol\left(\mathbb{C} \setminus \mathbb{R}_{\geq 0} \cup \{-i,+i\}\right)$$



So consider the path



and write

$$J(\alpha) := \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \left(\int_{C_R} F\,dz - \int_{C_\epsilon} F\,dz + \int_\epsilon^R \underbrace{F(x+i0)}_{=F(x)}\,dx - \int_\epsilon^R \underbrace{F(x-i0)}_{=F(x)e^{2\pi i\alpha}}\,dx \right)$$

$$\underset{\text{Res. thm.}}{=} 2\pi i \left(\operatorname{Res}_i (F(z)) + \operatorname{Res}_{-i}(F(z)) \right)$$

$$= 2\pi i \left(\operatorname{Res}_i \left(\frac{e^{\alpha \log z}/(z+i)}{z-i} \right) + \operatorname{Res}_{-i} \left(\frac{e^{\alpha\log z}/(z-i)}{z+i} \right) \right)$$

$$= 2\pi i \left(\frac{e^{\alpha \log i}}{2i} + \frac{e^{\alpha\log(-i)}}{-2i} \right)$$

$$= 2\pi \, \frac{e^{\alpha\frac{\pi i}{2}} - e^{\alpha\frac{3\pi i}{2}}}{2} = 2\pi i e^{\alpha\pi i} \cdot \frac{e^{-\alpha\pi i/2} - e^{\alpha\pi i/2}}{2i}$$

$$= -2\pi i e^{\alpha\pi i} \sin(\alpha\pi/2) .$$

On the other hand (noting $\alpha \in \mathbb{R}$)

$$|z^\alpha| = |e^{\alpha\log z}| \underset{z=Re^{i\theta}}{=} |e^{\alpha\log R}|\,\cancel{|e^{i\alpha\theta}|}\ 1$$

$$\left| \int_{C_R} F\,dz \right| \le 2\pi R \left\| \frac{z^\alpha}{1+z^2} \right\|_{C_R} \le 2\pi R \cdot \frac{R^\alpha}{R^2-1} \xrightarrow[(R\to\infty)]{\alpha+1<2} 0$$

$$\left| \int_{C_\epsilon} F\,dz \right| \le 2\pi\epsilon \left\| \frac{z^\alpha}{1+z^2} \right\|_{C_\epsilon} \le 2\pi\epsilon \cdot \frac{\epsilon^\alpha}{\underbrace{-\epsilon^2}_{0}+1} \xrightarrow[(\epsilon\to 0)]{\alpha+1>0} 0$$

So $J(\alpha) = 0 - 0 + \lim \int_\epsilon^R F\,dz - e^{2\pi i \alpha} \lim \int_\epsilon^R F\,dz$

$$= \left(1 - e^{2\pi i \alpha}\right) I(\alpha)$$

$$= 2ie^{\pi i \alpha}\, \frac{e^{-\pi i \alpha} - e^{\pi i \alpha}}{2i}\, I(\alpha)$$

$$= -2ie^{\alpha\pi i} \sin(\pi\alpha)\, I(\alpha)$$

and $-2\pi i e^{\alpha\pi i} \sin(\alpha\pi/2) = J(\alpha) = -2i e^{\alpha\pi i} \sin(\pi\alpha) I(\alpha)$

$$\Rightarrow \boxed{I(\alpha) = \frac{\pi \sin\left(\frac{\alpha\pi}{2}\right)}{\sin(\alpha\pi)}}.$$

More generally, Mellin transforms are almost always computed using the residues of the function $f(z) z^{k-1}$ in the complex plane (except $z=0$), and by the same technique.

IV. Application to modular forms

We'll jump into some deep water here; related stuff will be done more carefully later when we look at Gamma and Riemann zeta functions.

Let $g \in \mathcal{H}ol(\mathfrak{h})$ satisfy the "automorphy property"

$$(*)_k \quad \begin{cases} g(f_\gamma(\tau)) = (c\tau+d)^k g(\tau) \\ \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in SL_2(\mathbb{Z}) . \end{cases}$$

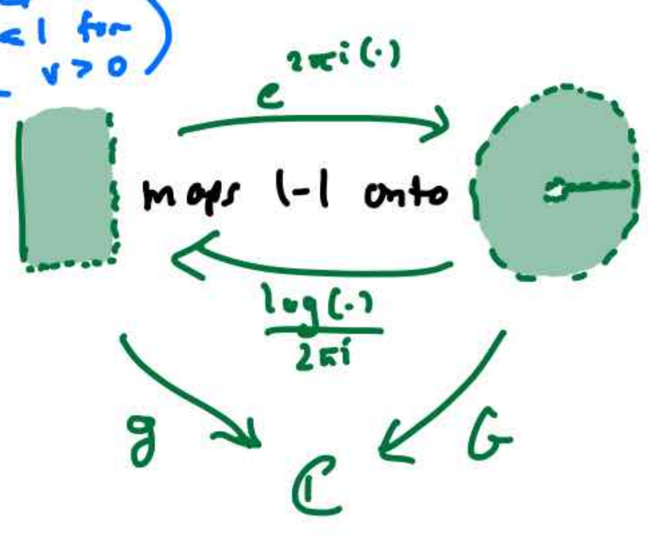
Since this is compatible with the group structure (HW) and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ generate $SL_2(\mathbb{Z})$ (HW), it is actually enough to check that

$$\left.\begin{array}{ll} (A) & g(\tau+1) = g(\tau) \\ (B) & g(-1/\tau) = \tau^k g(\tau) \end{array}\right\} (*)_k \text{ for}$$

From (A), $g(\tau) = G(e^{2\pi i \tau})$ for some $G \in \mathcal{H}ol(D_1^*)$:

$\tau \mapsto e^{2\pi i \tau}$ maps $\mathfrak{h}/\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle \xrightarrow{\cong} D_1^*$, since

$e^{2\pi i(u+iv)} = e^{2\pi i u} \underbrace{e^{-2\pi v}}_{(<1 \text{ for } v>0)}$. That is, each vertical strip



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Using Laurent series,

$$G(q) = \sum_{n=-\infty}^{\infty} a_n q^n \implies g(\tau) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\tau}.$$

Now

(1) 0 is a removable singularity of G $\iff$ $a_n = 0$ for $n < 0$ $\iff$ g "bounded at ∞" (i.e. on $\{\tau \mid \mathrm{Im}(\tau) \geq a\}$ for $a > 0$)

$\overset{\text{def.}}{\iff}$ g is a modular form.

(2) $G(0) = 0 \iff a_n = 0$ for $n \leq 0$ $\overset{\text{def.}}{\iff}$ g is a cusp form.

(We say g has weight k if $(*)_k$ holds.)

Suppose then that $g(\tau) = \sum_{n \geq 1} c_n e^{2\pi i n\tau}$ is a cusp form of weight k, and put (U_0 := right-half-plane)

$$\tilde{g}(z) := g(iz) \in \mathrm{Hol}(U_0).$$

Computing "formally", the Mellin transform

$$(M\tilde{g})(s) = \int_0^\infty \tilde{g}(x)\, x^{s-1}\, dx$$

$$= \int_0^\infty \sum_{n \geq 1} c_n e^{2\pi i n(ix)} x^{s-1}\, dx$$

$$= \sum_{n\geq 1} c_n \int_0^\infty e^{-2\pi n x} x^{s-1}\, dx$$

$\tau = 2\pi n x$

$$= \sum_{n\geq 1} c_n \int_0^\infty e^{-t} (2\pi n)^{-s} t^{s-1}\, dt$$

$$= (2\pi)^{-s}\Gamma(s) \sum_{n\geq 1} \frac{c_n}{n^s}$$

$$=: (2\pi)^{-s}\Gamma(s) \underline{L(s,g)}$$

$$=: L^*(s,g).$$ "L-function of the cusp form g"

So, for which s is this valid?

Consider the function

$$\varphi(\tau) := |g(\tau)|\, \mathrm{Im}(\tau)^{k/2} \in C^\infty(\mathfrak{h}).$$

This satisfies $\varphi(\tau+1) = \varphi(\tau)$ and

$$\varphi(-1/\tau) = |g(-1/\tau)| \left(\frac{\mathrm{Im}(\tau)}{|\tau|^2}\right)^{k/2}$$

$$= \cancel{|\tau|^k}\, |g(\tau)|\, \mathrm{Im}(\tau)^{k/2} \cancel{|\tau|^{-k}} = \varphi(\tau).$$

So it is $SL_2(\mathbb{Z})$-invariant.

<u>Lemma 1</u>: φ is bounded (on $\mathfrak{h}$).

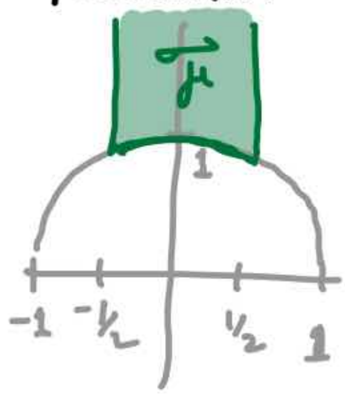
Sketch: $|G(q)| \leq C|q|$ for $|q| \leq \frac{1}{2}$

$$\Rightarrow |g(\tau)| \leq C e^{-2\pi \mathrm{Im}(\tau)} \text{ for } \mathrm{Im}(\tau) \geq \frac{\log 2}{2\pi}$$

$$\Rightarrow \varphi(\tau) \leq C e^{-2\pi \operatorname{Im}(\tau)} \operatorname{Im}(\tau)^{k/2} \to 0$$

$$(\text{as } \operatorname{Im}(\tau) \to \infty).$$

We'll learn later about the standard fundamental domain for $SL_2(\mathbb{Z})$, which is $\mathcal{F}$:



Clearly φ is bounded on $\mathcal{F}$, and so by invariance it's bounded on $\mathfrak{h}$. $\square$

<u>Lemma 2</u>: $|c_n| \leq C_0 \cdot n^{k/2}$.

Sketch:

$$c_n = \frac{1}{2\pi i} \oint_{\underbrace{\partial D_r}_{\text{disk of radius } r<1}} \frac{G(q)\,dq}{q^{n+1}} = \cancel{\frac{1}{2\pi i}} \underbrace{\int_{-1/2+ib}^{1/2+ib}}_{(b>0)} \frac{g(\tau)\,\cancel{2\pi i e^{2\pi i \tau}}}{e^{2\pi i (n+\cancel{1})\tau}}\,d\tau$$

$$= \int_{-1/2+ib}^{1/2+ib} g(\tau) e^{-2\pi i n \tau}\,d\tau$$

$$\Rightarrow |c_n| \leq \|g(\tau)\| e^{2\pi n b} \underset{\text{Lemma 1}}{\leq} C b^{-k/2} e^{2\pi n b} \quad (\forall b>0)$$

$$\Rightarrow |c_n| \underset{b = 1/n}{\leq} C \cdot n^{k/2} e^{2\pi} = C_0 n^{k/2}.$$

$\square$

cusp form of weight k

Proposition The series defining $L(s,g)$ converges absolutely for $\mathrm{Re}(s) > \frac{k}{2}+1$, hence the "formal" computation above is valid.

Sketch: $\frac{c_n}{n^s} \underset{\text{lemma 2}}{\leq} C \cdot \frac{1}{n^{s-k/2}}$, apply integral test;

then notice that in the $\sum\int \Leftrightarrow \int\sum$ swap, all functions / terms are > 0. $\square$

Theorem (Hecke) $L(s,g)$ extends to an entire function, and may be described in $\mathrm{Re}(s) < \frac{k}{2}-1$ by means of the functional equation

$$L^*(s,g) = (-1)^{k/2} L^*(k-s, g).$$

Proof: Write $\tau = u+iv$, $u=0$.

- $g(-1/\tau) = \tau^k g(\tau)$ becomes $g(i/v) \overset{(*)}{=} i^k v^k g(iv)$.
- $|g(\tau)| \leq C e^{-2\pi v}$ for $v \geq \frac{1}{2\pi}\log 2$ $\Longrightarrow$

$\int_1^\infty g(iv)\, v^{s-1}\, dv$ converges for all s (to an entire fcn.).

Now, for $\mathrm{Re}(s) > \frac{k}{2}+1$,

$$L^*(s,g) = \underbrace{\int_0^1 g(iv)\, v^{s-1}\, dv}_{\text{replace } v \text{ by } 1/v \,:\, -\int_1^\infty g\left(\frac{i}{v}\right) v^{1-s} \frac{dv}{v^2} \overset{(*)}{=} i^k \int_1^\infty v^{k-s-1} g(iv)\,dv} + \int_1^\infty g(iv)\, v^{s-1}\, dv$$

$$= \int_1^\infty g(iv) \left\{ v^{s-1} + i^k v^{k-s-1} \right\} dv \quad \in \mathrm{Hol}(\mathbb{C}).$$

replace s by $k-s$

Also, $i^k L^*(k-s, g) = i^k \int_1^\infty g(iv) \left\{ v^{k-s-1} + i^k v^{s-1} \right\} dv$

$= $ above, if k is even.

But there are no cusp forms of odd weight: if $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, then $g(\tau) = g(\underbrace{f_\gamma(\tau)}_{\text{(identity)}}) = (-1)^k g(\tau) \implies g \equiv 0$. $\square$

Example // When we discuss infinite products we'll meet Weierstraß's modular discriminant †

$$\Delta(\tau) = \sum_{n \geq 1} c_n q^n = q - 24q^2 + 252 q^3 - \ldots$$

usually called $\tau(n)$ (obviously a conflict of notation here), the Ramanujan tau function.

Properties :

† $\Delta = g_2^3 - 27 g_3^2$ is discriminant of $y^2 = 4x^3 - g_2 x - g_3$ (elliptic curve).

- Δ is a cusp form of weight 12
- $c_n c_m = c_{nm}$ if $\gcd(n,m) = 1$
- $|c_p| \le 2p^{11/2}$ for p prime (from above, know at least $|c_n| \le C_0 \cdot n^6$)

__Conjecture (Ramanujan):__ The zeros of $L(s,\Delta)$ in the critical strip $\frac{11}{2} < \mathrm{Re}(s) < \frac{13}{2}$ all lie on $\mathrm{Re}(s) = 6$.

(Remind you of something? Only, I suppose this one isn't worth \$1,000,000.) //