Lecture 28: An algebro-geometric detour

In this lecture I will give two important applications of Rouché + Riemann to complex algebraic curves.

I. Connectedness of irreducible algebraic curves

An affine algebraic curve $C \subset \mathbb{C}^2$ is the solution set of a polynomial equation

(1) $$0 = f(x,y) = y^n + a_1(x)y^{n-1} + \dots + a_n(x) \quad (\deg(a_i) \leq i)$$

where the $\{a_i(x)\}$ are polynomials (in x).† Here I am using x & y to denote two complex variables.

Projecting

$$\pi : C \twoheadrightarrow \mathbb{C}$$
$$(x,y) \mapsto x \quad ,$$

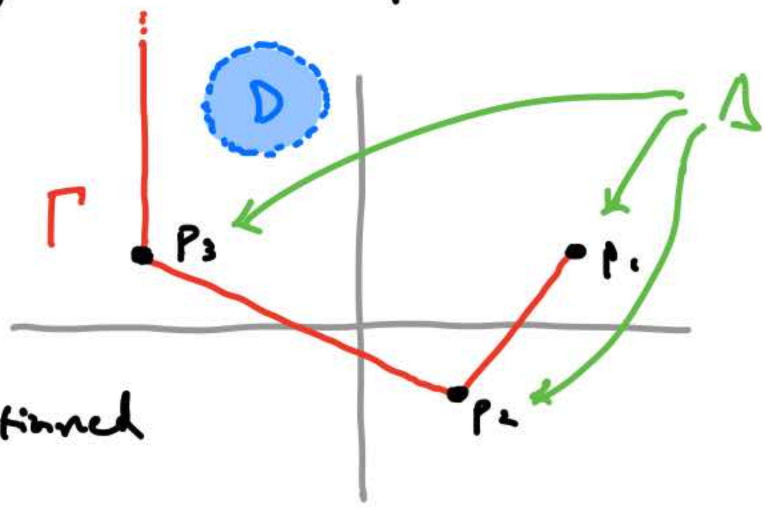
we find that on the complement of a finite point set $\Delta \subset \mathbb{C}$, the preimage consists of exactly n points:

(2) $$\pi^{-1}(x) = \{y_1(x), \dots, y_n(x)\}.$$

More precisely, one writes (2) on a small disk

† or rather, the equation can always be put in this form by a change of variable of the form $(x_0, y_0) \mapsto (x,y) = (x_0 - \lambda y_0, y_0)$.

$D \subset \mathbb{C}\setminus\Delta$ and analytically continues the resulting $\{y_i(x)\}$ to the simply connected complement of a set Γ (as shown):



even if continued through $\Gamma\setminus\Delta$, these analytically continued $\{y_i(x)\}$ still satisfy

$$f(x, y_i(x)) = 0 .$$

But: they may swap! (Think back to Riemann surfaces.)

$\boxed{\text{Proposition}}$ Let $E \subset \{1, \dots, n\}$ be a proper subset which is closed under this swapping (under continuation over all of $\mathbb{C}\setminus\Delta$), which we may assume for simplicity to be of the form $\{1, \dots, m\}$ $(m < n)$. Then $\prod_{l=1}^{m} (y - y_l(x))$ is a polynomial in x and y, and so (1) factors nontrivially in $\mathbb{C}[x,y]$.

$\boxed{\text{Theorem}}$ If C is irreducible (i.e. f is irreducible in $\mathbb{C}[x,y]$), then $C\setminus\pi^{-1}(\Delta)$ is connected (and therefore so is C).

Prop. ⇒ Thm.: If f is irred., then by the Proposition, the only subset of $\{1,\dots,n\}$ closed under the "swapping action" $\pi_1(\mathbb{C}\setminus\Delta) \to \mathcal{S}_n$ is $\{1,\dots,n\}$ itself. So the complete set of "branches" $\{y_i(x)\}$ is acted on transitively by "monodromy" about Δ, and one can therefore draw a path on $\mathcal{C}\setminus\pi^{-1}(\Delta)$ connecting any two points. $\square$

Proof of Prop.: $\prod_{\ell=1}^{m}(y-y_\ell(x))$ is clearly well-defined on $\mathbb{C}\setminus\Delta$, since continuation through $\mathbb{P}^1\setminus\Delta$ ("monodromy about Δ") merely swaps its factors. Thus it is in $\mathcal{H}ol(\mathbb{C}\setminus\Delta)$ for each fixed y, and we write

elem. symm. polynomial

$$\prod_{\ell=1}^{m}(y-y_\ell(x)) = \sum_{j=0}^{m} \underbrace{e_{m-j}\left(\{y_\ell(x)\}_{\ell=1}^{m}\right)}_{\text{again, monodromy-invariant hence in } \mathcal{H}ol(\mathbb{C}\setminus\Delta)\text{. But } \underline{\text{why polynomials}}\text{??}} \cdot y^j$$

Let $\alpha\in\Delta$, $\mathcal{N}_\alpha$ a small disk about α.

We have

$$x\in\mathcal{N}_\alpha \implies |a_j(x)| \le M \quad (\forall j).$$

Fix $x_0\in\mathcal{N}_\alpha^*$, put $a_j := a_j(x_0)$; then we shall apply Rouché to

$$\begin{cases} \tilde{\mathcal{Y}}(y) = y^n \\ \mathcal{Y}(y) = y^n + a_1 y^{n-1} + \cdots + a_n \\ \gamma = \{|y| = M+1\} \subset \mathbb{C} \\ |\tilde{\mathcal{Y}} - \mathcal{Y}| = |a_1 y^{n-1} + \cdots + a_n| \leq M((M+1)^{n-1} + \cdots + 1) \end{cases}$$

on γ

$$= (M+1)^n - 1$$

$$< (M+1)^n = |\tilde{\mathcal{Y}}|$$

$\Longrightarrow$ $\tilde{\mathcal{Y}}$ & $\mathcal{Y}$ have the same # of zeroes ($= n$ for $\tilde{\mathcal{Y}}$, hence for $\mathcal{Y}$) inside γ

$\Longrightarrow$ $|y_j(x_0)| < M+1$ ($\forall\, j = 1, \dots, n$ & $x_0 \in \mathcal{N}_\alpha^*$)

$\Longrightarrow$ the $e_k(\{y_1(x), \dots, y_m(x)\})$ are bounded on $\mathcal{N}_\alpha^*$ (as well as single-valued)

$\Longrightarrow$ extend to $\mathcal{N}_\alpha$.

Riemann

Conclude that the $e_k(\{y_1(x), \dots, y_m(x)\}) \in \mathcal{H}ol(\mathbb{C})$ (are entire).

To show they are polynomials, we've got to analyze their behavior <u>about ∞</u>. Change coordinates to $\tilde{x} = \frac{1}{x}$, $\tilde{y} = \frac{y}{x}$: (1) becomes

$$0 = \tilde{x}^n f\left(\tfrac{1}{\tilde{x}}, \tfrac{\tilde{y}}{\tilde{x}}\right) = \tilde{y}^n + \underbrace{\left(\tilde{x}\, a_1\left(\tfrac{1}{\tilde{x}}\right)\right)} \tilde{y}^{n-1} + \dots + \underbrace{\tilde{x}^n a_n\left(\tfrac{1}{\tilde{x}}\right)},$$

(polynomials in $\tilde{x}$)

with roots $$\tilde{y}_i(\tilde{x}) = \tilde{x} \cdot y_i\left(\tfrac{1}{\tilde{x}}\right).$$

From this formula, clearly the $\{\tilde{y}_i\}_{i=1}^m$ are permuted amongst themselves by monodromy. So the above argument applies to $e_k(\{\tilde{y}_1(\tilde{x}), \dots, \tilde{y}_m(\tilde{x})\})$ on a neighborhood $\mathcal{N}_\infty$; that is,

$$\mathrm{Hol}(\mathcal{N}_\infty) \ni e_k\left(\{\tilde{y}_\ell(\tilde{x})\}_{\ell=1}^m\right) = \tilde{x}^k e_k\left(\left\{y_\ell\left(\tfrac{1}{\tilde{x}}\right)\right\}_{\ell=1}^m\right) = \frac{e_k\left(\{y_\ell(x)\}_{\ell=1}^m\right)}{x^k}$$

and so $e_k(\{y_\ell(x)\}_{\ell=1}^m)$ has at worst a pole of order k at ∞ (and is holo. on $\mathbb{C}$). Therefore it is a polynomial of degree at most k (in x). $\square$

II. Meromorphic functions on algebraic curves are algebraic (rational)

Starting with the same projection (taking C now to be irreducible)

$$\pi : C \twoheadrightarrow \mathbb{C},$$

we have inclusions of fields

$$\pi^* \mathbb{C}(x) \subset \underbrace{\mathbb{C}(C)}_{} \subset \mathcal{M}er(\overline{C}) .$$

(rational functions in x, y, restricted to C)

(closure in $\mathbb{P}^2$ [compactification], assumed smooth)

Claim: (a) $[\mathbb{C}(C) : \pi^* \mathbb{C}(x)] \geq n$

(b) $[\mathcal{M}er(\overline{C}) : \pi^* \mathbb{C}(x)] \leq n$

$\Downarrow$

$\boxed{\text{Theorem}}$ $\underline{\mathcal{M}er(\overline{C}) = \mathbb{C}(C)}$.

(a) is trivial: by irreducibility of C,

$$0 = f(x,y) = y^n + a_1(x) y^{n-1} + \dots + a_n(x)$$

is the minimal polynomial of y over $\pi^* \mathbb{C}(x)$.

More interesting is the

Proof of (b): Let $\varphi \in \mathcal{M}er(\overline{C})^*$, and denote by P the image under π of its polar set and $\tilde{\Delta} = \Delta \cup P$. Write

$$e_i^{\varphi}(x) := e_i\left(\{\varphi(x, y_1(x)), \dots, \varphi(x, y_n(x))\}\right) \in \mathcal{H}ol(\mathbb{C} \setminus \tilde{\Delta}).$$

The same argument as above involving Riemann + Rouché

$\implies$ e_i^{φ} extend to holomorphic functions on $\mathbb{C} \setminus P$.

About a point $\beta \in P$, we can look at e_i^{φ} times a sufficiently high power of $(x-\beta)$; this cancels out the pole of φ at $(\beta, \dots)$ and allows the same argument to be applied, with the consequence that e_i^φ has <u>meromorphic</u> extension to $\mathbb{C}$. Again, analyzing about $\infty \Rightarrow e_i^\varphi \in \mathcal{M}er(\mathbb{P}^1) \cong \mathbb{C}(x)$.

Now for any $\alpha \in \mathbb{C} \setminus \Gamma \cup P$ and $j \in \{1, \dots, n\}$,

$$0 = \prod_{i=1}^{n} \left(\varphi(x, y_j(x)) - \varphi(x, y_i(x)) \right)$$

$$= \sum_{k=0}^{n} \varphi(x, y_j(x))^{n-k} \cdot (-1)^k e_k^\varphi(x)$$

$\Rightarrow$ the meromorphic function φ itself satisfies

$$0 = \sum_{k=0}^{n} (-1)^k (\pi^* e_k^\varphi) \cdot \varphi^{n-k}$$

$\Rightarrow$ any element of $\mathcal{M}er(\overline{C})^*$ has degree $\leq n$ over $\pi^* \mathbb{C}(x)$

$\Rightarrow$ the degree of the extension is no more than n.

(primitive element theorem!)

(Otherwise, there is a primitive element, of degree $> n$.)

$\square$