Lecture 27: Computing real integrals

The most elementary motivation for residue calculus is to apply path integrals and residues in $\mathbb{C}$ to the computation of real integrals.

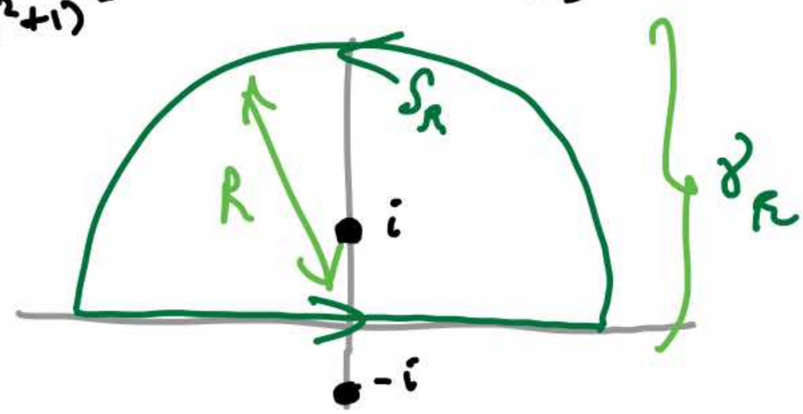
I. Rational integrals

We saw the example that follows in Lecture 25, but I'd like to review it to illustrate a more general principle.

Example // $$I = \int_0^\infty \frac{x^2}{(1+x^2)^2}\,dx$$

$$\Rightarrow 2I = \int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{(1+x^2)^2}\,dx$$

Set $f(z) := \frac{z^2}{(z^2+1)^2} \in \mathrm{Hol}(\mathbb{C}\setminus\{i,-i\})$, and consider the path



Now from $|z^2+1| \geq |z|^2 - 1 = R^2 - 1$ (on S_R)

$$\Rightarrow \frac{|z|^2}{|z^2+1|^2} \leq \frac{R^2}{(R^2-1)^2} \quad (\text{on } S_R)$$

We have

$$\left| \int_{S_R} f\, dz \right| \leq L(S_R) \|f\|_{S_R} \leq \pi R \cdot \frac{R^2}{(R^2-1)^2} = \frac{\pi R^3}{R^4 - 2R^2 + 1} \longrightarrow 0$$

(as $R \to \infty$),

so

$$2I = \lim_{R\to\infty} \left(\int_{\gamma_R} f\, dz - \int_{S_R} f\, dz \right)$$

$$= 2\pi i \operatorname{Res}_i(f) - 0$$

$$= 2\pi i \operatorname{Res}_i(f).$$

Writing $g(z) = \dfrac{z^2}{(z+i)^2}$ $\left(\Rightarrow f(z) = \dfrac{g(z)}{(z-i)^2} \right)$,

$$g'(z) = \frac{2z(z+i)^{\cancel{2}} - 2z^2 \cancel{(z+i)}}{(z+i)^{\cancel{4}\,3}} = \frac{2iz}{(z+i)^3}$$

$$g'(i) = \frac{-2}{(2i)^3} = \frac{\cancel{-2}}{\cancel{-8}_{4}\, i} = \frac{-i}{4},$$

we have

$$I = \pi i \operatorname{Res}_i(f) = \pi i \cdot \frac{g'(i)}{(2-1)!} = \pi i \cdot \left(-\frac{i}{4} \right) = \frac{\pi}{4},$$

which is real and positive (as expected). //

Exactly the same reasoning leads to the following result; note that the $\int_{-\infty}^{\infty}$ in question converges by the assumption on degrees.

Theorem Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function with $\deg(Q) \geq 2 + \deg(P)$, and all roots of Q non-real. Then if $z_j = x_j + i y_j$ are the roots of $Q(z)$, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{j:\, y_j > 0} \operatorname{Res}_{z_j}(f).$$

Remark:

More generally, the result holds provided $f(x)$ is the restriction to $\mathbb{R}$ of $f(z) \in \operatorname{Mer}(\overline{h})$ with only finitely many poles (in h) and none on $\mathbb{R}$, and if $\exists\, a > 1$ s.t.

$$|f(z)| \leq B/_{|z|^a} \text{ for } |z| \geq K.$$

//

"Proof": We only need to show that the integral over the big semicircle vanishes as $R \to \infty$:

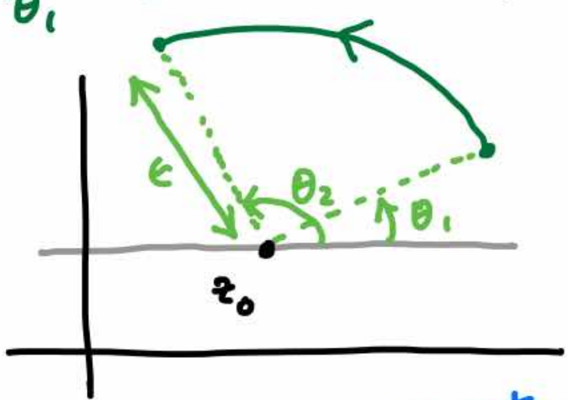
$$\lim_{R\to\infty} \left| \int_{S_R} f(z)\,dz \right| \le \lim_{R\to\infty} \pi R \cdot \frac{B}{R^a} = \lim_{R\to\infty} \frac{\pi B}{R^{a-1}} = 0. \qquad \square$$

II. How to deal with a pole† on $\mathbb{R}$

Consider the integral

$$I = \int_0^\infty \frac{\sin(x)}{x}\,dx.$$

Lemma: If $f \in \mathrm{Hol}(\bar{D}^*(z_0, r))$ has a simple pole at z_0, and $A_{\theta_1}^{\theta_2}(z_0, \epsilon)$ is the path (arc)



then

$$\lim_{\epsilon\to 0} \int_{A_{\theta_1}^{\theta_2}(z_0,\epsilon)} f(z)\,dz = i\underbrace{(\theta_2 - \theta_1)}_{\text{length of arc in radians}} \operatorname{Res}_{z_0}(f).$$

† $\frac{\sin(z)}{z}$ doesn't have one, but bear with me...

Proof: $f(z) = \dfrac{\mathrm{Res}_{z_0}(f)}{z - z_0} + h(z)$, $h \in \mathrm{Hol}(\bar{D})$, $|h| \leq B$ on $\bar{D}$

$$\Rightarrow \int_{A_{\theta_1}^{\theta_2}(z_0,\epsilon)} f(z)\,dz \underset{\uparrow}{=} \int_{\theta_1}^{\theta_2} f(\gamma(\theta))\,\gamma'(\theta)\,d\theta$$

$\gamma(\theta) = \epsilon e^{i\theta} + z_0$

$$= \int_{\theta_1}^{\theta_2} \frac{\mathrm{Res}_{z_0}(f)}{\cancel{\epsilon e^{i\theta}}}\, i\cancel{\epsilon e^{i\theta}}\,d\theta + \int_{\theta_1}^{\theta_2} h(\epsilon e^{i\theta})\, i\epsilon e^{i\theta}\,d\theta$$

$$= i(\theta_2 - \theta_1)\,\mathrm{Res}_{z_0}(f) + \underbrace{i\epsilon \int_{\theta_1}^{\theta_2} e^{i\theta} h(\epsilon e^{i\theta})\,d\theta}_{|\;\text{''}\;| \leq \epsilon B(\theta_2 - \theta_1) \to 0 \text{ (as } \epsilon \to 0)}$$

$\square$

So $$2I \underset{\uparrow}{=} \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx = \lim_{\epsilon \to 0}\left[\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right)\frac{\sin x}{x}\,dx\right]$$

$\left(\frac{\sin(x)}{x} \text{ even}\right)$

$\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right)\frac{\cos x}{x}\,dx = 0$, and $\frac{\sin x}{x} = -\frac{ie^{ix}}{x} + \frac{i\cos x}{x}$

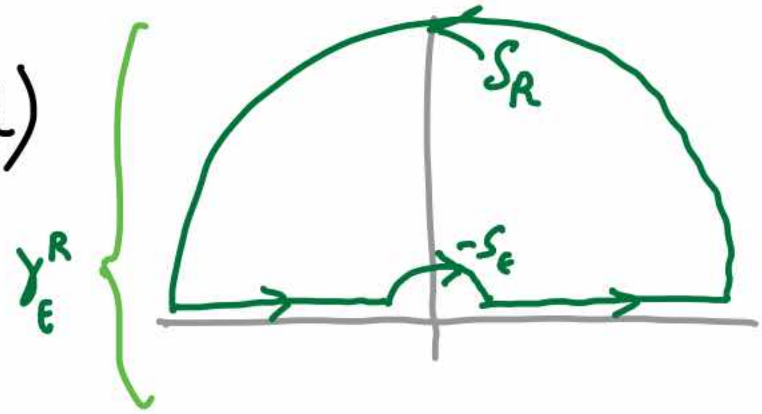
$$= \lim_{\epsilon \to 0}\left[\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right)\frac{-ie^{ix}}{x}\,dx\right]$$

$$=: -i \cdot \mathrm{P.V.}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx\right).$$

this is called the P.V. (="principal value") of an $\int$ even if the $\int$ itself isn't well-def'd

Now $\frac{e^{iz}}{z}$ has no poles in $\mathfrak{h}$, so

$$0 = \int_{\gamma_\epsilon^R} \frac{e^{iz}}{z}\,dz \quad (\forall \epsilon, R)$$



$$= \lim_{\epsilon\to 0}\lim_{R\to\infty}\left\{\left(\int_{S_R} + \int_{-S_\epsilon} + \int_{-R}^{-\epsilon} + \int_{\epsilon}^{R}\right)\frac{e^{iz}}{z}\,dz\right\}$$

$$= \underbrace{\lim_{R\to\infty}\int_{S_R}\frac{e^{iz}}{z}\,dz}_{=:\ \mathrm{II}} + \underbrace{\lim_{\epsilon\to 0}\int_{-S_\epsilon}\frac{e^{iz}}{z}\,dz}_{=:\ \mathrm{III}} + \mathrm{P.V.}\left[\int_{-\infty}^{\infty}\frac{e^{iz}}{z}\,dz\right]$$

and

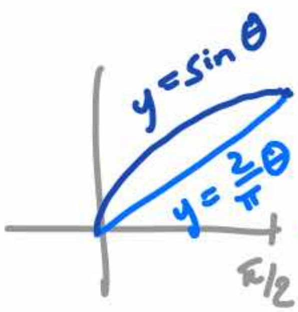
$$\mathrm{III} = i(0-\pi)\underbrace{\mathrm{Res}_0\left(\frac{e^{iz}}{z}\right)}_{1} = -\pi i \qquad \text{(by Lemma)}$$

$\mathrm{II} = 0$ since

$$\left|\int_{S_R}\frac{e^{iz}}{z}\,dz\right| = \left|\int_0^{\pi}\frac{e^{iR\cos\theta}\,e^{-R\sin\theta}}{\cancel{\gamma(\theta)}}\,i\,\cancel{\gamma(\theta)}\,d\theta\right|$$

$$\begin{cases}\gamma(\theta) = Re^{i\theta} = R(\cos\theta + i\sin\theta)\\ \gamma'(\theta) = Rie^{i\theta} = i\gamma(\theta)\end{cases}$$

$$\leq \int_0^{\pi} e^{-R\sin\theta}\,d\theta$$



$$= 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta$$

$$\leq 2\int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta$$

$$= 2\left[-\frac{\pi}{2R}e^{-2R\theta/\pi}\right]_0^{\pi/2}$$

$$= \frac{\pi}{R}\left(1-e^{-R}\right) \longrightarrow 0 \text{ as } R\to\infty .$$

Hence, $\text{P.V.}\left(\int_{-\infty}^{\infty}\frac{e^{iz}}{z}dz\right) = -\text{III} = \pi i$

and $I = \frac{-i}{2}\cdot \text{P.V.}\left(\int_{-\infty}^{\infty}\frac{e^{iz}}{z}dz\right) = \frac{-i}{2}\cdot\pi i = \frac{\pi}{2}.$

III. Trigonometric integrals

Let $R(x,y)$ be a rational function in x & y, and consider

$$\mathcal{I} := \int_0^{2\pi} R(\cos\theta,\sin\theta)\,d\theta .$$

We will make the substitution

$$z = e^{i\theta}$$

$$dz = ie^{i\theta}d\theta = iz\,d\theta$$

$$-i\frac{dz}{z} = d\theta$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

to rewrite

$$\mathcal{I} = -i\oint_{C_1}\overbrace{R\left(\frac{1}{2}\left[z+\frac{1}{z}\right],\frac{1}{2i}\left[z-\frac{1}{z}\right]\right)\frac{dz}{z}}^{=:f(z)}$$

$$= 2\pi i\left(-i\sum_{p\in D_1}\text{Res}_p(f(z))\right) = 2\pi\sum_{p\in D_1}\text{Res}_p(f(z)).$$

(Of course, one should assume that the original integral was not improper.)

Example// $I = \int_0^{\pi} \frac{d\theta}{a+\cos\theta}$, $a > 1$ (real).

$\Rightarrow \mathcal{I} = 2I = \int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$, with $R(x,y) = \frac{1}{a+x}$.

So $f(z) = \frac{1}{z} \cdot \frac{1}{a + \frac{1}{2}(z+\frac{1}{z})} = \frac{1}{az + \frac{1}{2}z^2 + \frac{1}{2}} = \frac{2}{z^2+2az+1}$

has (simple) poles at $z_{\pm} = \frac{-2a \pm \sqrt{4a^2-4}}{2} = -a \pm \sqrt{a^2-1}$.

Clearly $|z_-| = a + \sqrt{a^2-1} > 1$, while

$$
\begin{aligned}
2 < 2a &\Rightarrow a^2 - 2a + 1 < a^2 - 1 \\
&\Rightarrow a - 1 < \sqrt{a^2-1} \\
&\Rightarrow |z_+| = a - \sqrt{a^2-1} < 1 \\
&\Rightarrow z_+ \in D_1 , \ z_- \in D_1^c .
\end{aligned}
$$

Now $\mathrm{Res}_{z_+}(f) = \mathrm{Res}_{z_+}\left(\frac{2/(z-z_-)}{z-z_+}\right) = \frac{2}{z_+-z_-} = \cancel{\frac{2}{2\sqrt{a^2-1}}}$

$= \frac{1}{\sqrt{a^2-1}}$,

hence $\mathcal{I} = 2\pi \,\mathrm{Res}_{z_+}(f) = \frac{2\pi}{\sqrt{a^2-1}}$

$\Rightarrow I = \frac{\mathcal{I}}{2} = \frac{\pi}{\sqrt{a^2-1}}$. //