# Lecture 26: Rouché's theorem

## I. Counting zeroes

Recall that a closed path $\gamma \subset \mathbb{C}$ is said to "have an interior" if $W(\gamma, \alpha) = 0$ or $1$ ($\forall \alpha \in \mathbb{C} \setminus \gamma$); and that then $\mathrm{Int}(\gamma) := \{\alpha \in \mathbb{C} \setminus \gamma \mid W(\gamma, \alpha) = 1\}$.

For such a $\gamma$, let $U$ be an open set containing $\overline{\mathrm{Int}(\gamma)} = \mathrm{Int}(\gamma) \cup \gamma$, and $f \in \mathrm{Mer}(U)$ have no poles or zeroes on $\gamma$ itself. Then we had

Argument principle (AP): $\displaystyle \frac{1}{2\pi i} \int_\gamma \frac{f'}{f}\, dz = \sum_{p \in \mathrm{Int}(\gamma)} \mathrm{ord}_p(f)$

Generalized argument principle (GAP): given also $g \in \mathrm{Hol}(U)$,

$$\frac{1}{2\pi i} \int_\gamma g \frac{f'}{f}\, dz = \sum_{p \in \mathrm{Int}(\gamma)} g(p)\, \mathrm{ord}_p(f).$$

Of course, $\sum \mathrm{ord}_p(f)$ has the standard interpretation as (# of zeroes w/mult.) − (# of poles w/mult.).

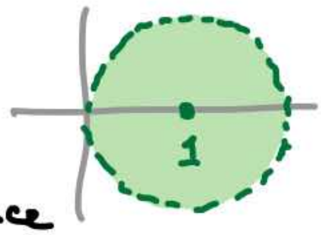
The following application of the <u>AP</u> is very useful:

$\boxed{\text{Rouché's Theorem}}$ <u>If $\gamma$ has an interior, $\overline{\mathrm{Int}(\gamma)} \subset U$ ($\Rightarrow \gamma \underset{\text{hom}}{\equiv} 0$ on $U$), and $f, g \in \mathcal{H}ol(U)$ satisfy</u>

$$(*) \qquad \boxed{|f-g| < |f| \text{ on } \gamma},$$

<u>then $f$ & $g$ have the same number of zeroes (counted w/multiplicity) in $\mathrm{Int}(\gamma)$.</u>

**Proof:** It follows at once from $(*)$ that on $\gamma$ $f$ & $g$ are nowhere zero and $\left|\frac{g}{f} - 1\right| < 1$. That is, the values of $g/f$ lie in $D(1,1)$.




It follows that $W\left(\frac{g}{f}\circ\gamma, 0\right) = 0$, hence

$$0 = \frac{1}{2\pi i}\int_{\frac{f}{g}\circ\gamma} d\log(z) = \frac{1}{2\pi i}\int_\gamma d\log\left(\frac{f}{g}\right)$$

$$= \frac{1}{2\pi i}\int_\gamma d\log(f) - \frac{1}{2\pi i}\int_\gamma d\log(g)$$

$$= \begin{pmatrix}\#\text{ of zeroes of } f \\ \text{counted w/mult.}\end{pmatrix} - \begin{pmatrix}\#\text{ of zeroes of } g \\ \text{counted w/mult.}\end{pmatrix}. \qquad \square$$

**Example** // Consider a polynomial with one BIG coefficient and the others small:

$$P(z) = a_d z^d + \cdots + a_1 z + a_0 + A z^m, \quad |A| > \sum |a_i|,$$

where $0 \le m \le d$. How many zeros does $P$ have inside the unit circle?

Set $g(z) := P(z)$, $f(z) := Az^m$, so that

$g(z) - f(z) = \sum a_j z^j$. On $C_1$,

$$|g - f| \le \sum |a_j| \underset{1}{\cancel{|z|^j}} < |A| = |f|.$$

Rouché $\Rightarrow$ $f$ & $g$ have same # of 0's w/mult. in $D_1$.

Since $f$ has one zero of multiplicity $m$ at $z = 0$, conclude that

<u>$P(z)$ has $m$ zeroes (counted w/mult.) inside the unit disk.</u> //

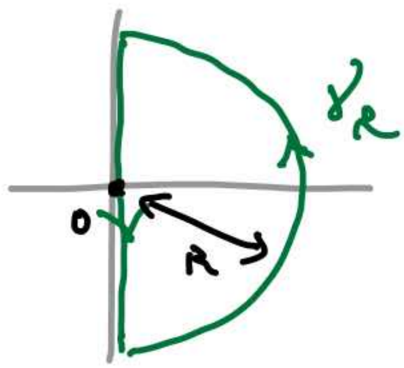
**Example**// How many solutions does

$$z^{2n+1} + e^{-z} = \lambda \;, \quad \lambda \in \mathbb{R}_{>1} \,,$$

have in the right half-plane?

Set $\left.\begin{array}{l} f(z) = z^{2n+1} - \lambda \\ g(z) = z^{2n+1} + e^{-z} - \lambda \end{array}\right\} \Rightarrow f - g = -e^{-z}$

and consider their behavior




on $\gamma_R$: we have

$$\begin{cases} |f-g| = |e^{-z}| = \frac{1}{e^x} \le 1 \\ \text{and} \\ \underbrace{|f| \ge \lambda}_{} \; (>1) \quad \text{there.} \end{cases}$$

To see this: • $z \in i\mathbb{R} \Rightarrow z^{2n+1} \in i\mathbb{R} \Rightarrow |z^{2n+1} - \lambda| = \sqrt{\lambda^2 + |z^{2n+1}|^2} \ge \lambda$

• $z \in$ semicircle $\Rightarrow$ can take $R$ sufficiently large that $|z|^{2n+1}$ exceeds $2\lambda$.

Taking $R \to \infty$,

Rouché $\Rightarrow$ $f, g$ have the same # of zeroes (w/mult.) in the right half-plane.

$f = 0 \iff z = \sqrt[2n+1]{\lambda} \cdot \zeta_{2n+1}^{j}$, $\begin{cases} n \; (n \text{ odd}) \\ n+1 \; (n \text{ even}) \end{cases}$ of which lie in the right half-plane.

So $g$ has $n$ resp. $n+1$ zeroes there too. //

# II. Inverse functions

Rouché's theorem provides a neat proof of the IMT:

$F \in \mathcal{H}ol(U)$, $F'(z_0) \neq 0 \Rightarrow F$ is a local analytic isomorphism at $z_0$.

- Assume $z_0 = 0$, write $F(z) = z + z^2 h(z)$, with $h \in \mathcal{H}ol(U)$ bounded on some closed disk: $|h(z)| \leq K$ for $z \in \overline{D_R}$.

- Take $\epsilon < \frac{1}{2K}$, let $|w| < \epsilon/2$; and set

$$g(z) := F(z) - w, \quad f(z) := z - w.$$

- For $z \in C_\epsilon$, $|f| = |z-w| \geq |z| - |w| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$; and so $|g - f| = |F - z| \leq K|z|^2 = K\epsilon^2 < \frac{\epsilon}{2} < |f|$.

- By Rouché, $g$ & $f$ have the same # of zeroes (w/mult.) in $D_\epsilon = \mathrm{Int}(C_\epsilon)$; $f$ has one at $z = w$, so $g$ has exactly one (w/mult. = 1).

Conclusion: For $|w| < \epsilon/2$ (i.e. sufficiently small), $\exists!$ $z \in D_\epsilon$ s.t. $F(z) = w$, giving an inverse

$$F^{-1} : D_{\epsilon/2} \to D_\epsilon$$

$$w \longmapsto F^{-1}(w) = z.$$

- Is this $F^{-1}$ continuous? holomorphic? — need this to conclude that $F$ is a local analytic isomorphism.

Use the GAP (now with $f = F(z) - w$ and $g = z$)

$$\underbrace{\frac{1}{2\pi i}\int_{C_\epsilon} \frac{F'(z)}{F(z)-w}\, z\, dz}_{} = \underbrace{\sum_{p \in D_\epsilon (= \mathrm{Int}(C_\epsilon))} \mathrm{ord}_p (F(z)-w) \cdot z(p)} = F^{-1}(w).$$

$\exists !\ p$ s.t. $F(p) - w = 0$ (& s.t. $\mathrm{ord}_p \neq 0$), namely $F^{-1}(w)$ — i.e. $z(\text{this } p) = F^{-1}(w)$

Such representations are holomorphic as a rule.

# III. Weierstraß factorizations

Similarly, the GAP gives us the analyticity of symmetric polynomials in the values of "multivalued inverses" — more important than you might think, as we'll see in the next lecture.

Assume

$$f(z) = w, \quad |w - w_0| < \delta$$

has $n$ solutions in $D(z_0, \epsilon)$. By the GAP with $g(z) = z^m$,

$$\underbrace{\frac{1}{2\pi i}\int_{\partial D(z_0,\epsilon)} z^m \frac{f'(z)}{f(z)-w}\,dz}_{\in\ \mathcal{H}ol(D(w_0,\delta))} = \sum_{j=1}^{n} (z_j(w))^m =: \underbrace{p_m(\{z_j(w)\})}_{\text{$m$th Newton symmetric polynomial in the } \{z_j\}}$$

[in particular, single-valued]

(In §II we basically saw the $m=1=n$ case.)

Formally writing (on $[(z,w)\in)$ $D(z_0,\epsilon)\times D(w_0,\delta)$)

$$f(z)-w = \underbrace{H(z,w)}_{\text{holo in } z,w;\ \text{nowhere vanishing}}\cdot \prod_{j=1}^{n}(z-z_j(w))$$

sum of $z_j$'s

(**) $$= H(z,w)\cdot\left\{z^n - e_1(\{z_j(w)\})z^{n-1} + e_2(\{z_j(w)\})z^{n-2} - \dots + (-1)^n e_n(\{z_j(w)\})\right\}$$

product of $z_j$'s.

defines the elementary symmetric polynomials in the $\{z_j(w)\}$. A well-known fact from algebra is that the $\{e_k\}$ are linear combinations of the $\{p_\ell\}$. Hence they are also <u>single-valued</u> and <u>holomorphic</u>, and we may replace "$f(z)=w$" by

$$0 = z^n - E_1(w)z^{n-1} + E_2(w)z^{n-2} + \dots + (-1)^n E_n(w).$$

The factorization (**) is called a <u>Weierstraß factorization</u>.

# IV. Bergman kernel on the disk

As a concluding application of residue theory, we will derive the first case of a "reproducing kernel" (in the setting of several complex variables) on a bounded symmetric domain (which in this case will just be $D_1$).

Writing $w \in D_1$, $f \in \mathrm{Hol}(D_1)$ (bounded on $D_1$), and

$$K(z,w) := \frac{i}{2\pi} \cdot \frac{1}{(1-\bar{z}w)^2},$$

we will show

$$(\dagger) \quad \boxed{f(w) = \iint_{z \in D_1} K(z,w)\, f(z)\, dz \wedge d\bar{z}}.$$

First note that $dz \wedge d\bar{z} = (dx + i\,dy) \wedge (dx - i\,dy) = -2i\, dx \wedge dy = -2i\, dA = -2i\, r\,dr\,d\theta$, and so

$$\mathrm{RHS}(\dagger) = \frac{1}{\pi} \iint_{|z|<1} \frac{f(z)\, d\theta \cdot r\,dr}{(1-\bar{z}w)^2}$$

$z = re^{i\theta} \Rightarrow d\theta = \frac{1}{i}\frac{dz}{z}$

$$= \frac{1}{i\pi} \int_0^1 \left( \int_{|z|=r} \frac{f(z)\, dz}{z(1-\bar{z}w)^2} \right) r\,dr$$

$$= \frac{1}{i\pi}\int_0^1 \left( \int_{|z|=r} \frac{z f(z)\,dz}{(z-r^2w)^2} \right) r\,dr$$

$$\frac{1}{2\pi i}\oint_{C_\varepsilon} \frac{F(z)}{z^2}\,dz = F'(0)$$

$$= \frac{2\cancel{\pi i}}{\cancel{\pi i}} \int_0^1 \Big[ \underbrace{f(r^2w) + r^2 w f'(r^2w)}_{(zf(z))'\big|_{z=r^2w}} \Big]\, r\,dr$$

$$= 2\int_0^1 \left\{ r f(r^2w) + r^3 w f'(r^2w) \right\} dr$$

$$= \Big[ r^2 f(r^2w) \Big]_0^1$$

$$= f(w), \quad \text{proving } (\dagger).$$

In general the Bergman kernel gives rise, by taking $\partial\bar\partial \log K(z,z)$, to a Hermitian metric.

What is the metric in this case?

$$\frac{\partial}{\partial \bar z} \log K(z,z) = -2\frac{\partial}{\partial \bar z}\log(1-z\bar z) = \frac{2z}{1-z\bar z}$$

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}\log K(z,z) = \frac{\partial}{\partial z}\frac{2z}{1-z\bar z} = 2\,\frac{(1-z\bar z)\cdot 1 - z(-\bar z)}{(1-z\bar z)^2} = \frac{2}{(1-|z|^2)^2}.$$

So $\partial\bar\partial \log K(z,z) = \dfrac{2\,dz\wedge d\bar z}{(1-|z|^2)^2}$, and replacing the antisym. (wedge) product by symmetric product $dz\cdot d\bar z$ gives

$$\frac{dz\otimes d\bar z + d\bar z\otimes dz}{(1-|z|^2)^2},$$

the Poincaré metric! (We'll find out why later.)