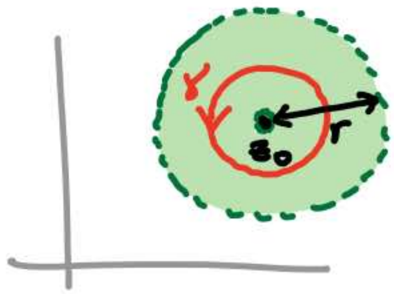
Lecture 25: Residue Calculus

I. Residues of functions

Recall the setup for studying isolated singularities:

$f \in \mathcal{H}ol(D^*(z_0, r))$



$\Rightarrow f(z) = \sum_{n \in \mathbb{Z}} a_n (z-z_0)^n.$

Define $\boxed{\mathrm{Res}_{z_0}(f) := a_{-1}}$.

<u>Fact</u>: $\int_\gamma f(z)\,dz = 2\pi i \,\mathrm{Res}_{z_0}(f)$

$$\int_\gamma \sum_{n\in\mathbb{Z}} a_n (z-z_0)^n dz = \sum_{n \in \mathbb{Z}} a_n \int_\gamma (z-z_0)^n dz = 2\pi i\, a_{-1}$$

<u>Remark</u>: f has a primitive on $D^* \iff \mathrm{Res}_{z_0}(f) = 0$.

RULES FOR COMPUTING RESIDUES

(0) Brute force with Laurent series

Ex// $\mathrm{Res}_0\left(\frac{e^z}{\sin^2 z}\right) = 1$:

$$\frac{e^z}{\sin^2 z} = \frac{1+z+\frac{z^2}{2}+\cdots}{\left(z-z^3/6+\cdots\right)^2} = \frac{1+z+\frac{z^2}{2}+\cdots}{z^2\left(1-\frac{z^2}{3}+\cdots\right)}$$

$$= \left(\frac{1}{z^2}+\frac{1}{z}+\frac{1}{2}+\cdots\right)\left(1+\frac{z^2}{3}-\cdots\right) = \frac{1}{z^2}+\frac{1}{z}+\cdots \quad //$$

Ex// $\mathrm{Res}_2\left(\frac{1}{z(z-2)^2}\right) = -\frac{1}{4}$:

$$\frac{1}{z(z-2)^2} = \frac{1}{z-2+2}\cdot\frac{1}{(z-2)^2} = \frac{1}{2}\,\frac{1}{1+\frac{z-2}{2}}\cdot\frac{1}{(z-2)^2}$$

$$= \frac{\frac{1}{2}\left(1-\frac{z-2}{2}+\frac{(z-2)^2}{4}-\cdots\right)}{(z-2)^2} = \frac{1}{2}(z-2)^{-2}-\frac{1}{4}(z-2)^{-1}+\cdots \quad //$$

There's a better method for the 2^{nd} example.
In fact, here are six:

① f holomorphic at $z_0 \Longrightarrow \mathrm{Res}_{z_0}\left(\frac{f(z)}{z-z_0}\right) = f(z_0)$

Ex// $\mathrm{Res}_\pi\left(\frac{\log(z)}{z-\pi}\right) = \log\pi$ //

Proof: $f(z) = b_0 + b_1(z-z_0)+\cdots$, where $b_0 = f(z_0)$

$$\frac{f(z)}{z-z_0} = \frac{b_0}{z-z_0} + b_1 + \cdots \qquad \square$$

② f holomorphic at $z_0 \Longrightarrow \mathrm{Res}_{z_0}\left(\frac{f(z)}{(z-z_0)^n}\right) = \frac{f^{(n-1)}(z_0)}{(n-1)!}$

Ex// $\mathrm{Res}_0\left(\frac{\sin z}{z^{10}}\right) = \frac{\left[\left(\frac{d}{dz}\right)^9 \sin z\right]_0}{9!} = \frac{\cos 0}{9!} = \frac{1}{9!}$ //

Ex// $\mathrm{Res}_2\left(\frac{1}{z(z-2)^2}\right) = \frac{-1/2^2}{1!} = -\frac{1}{4}$ (using $\frac{d}{dz}\frac{1}{z} = \frac{-1}{z^2}$) //

Proof: $f(z) = b_0 + b_1(z-z_0) + \dots + b_{n-1}(z-z_0)^{n-1} + \dots$

$$= f(z_0) + f'(z_0)(z-z_0) + \dots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1} + \dots$$

$$\frac{f(z)}{(z-z_0)^n} = \dots + \underbrace{\frac{f^{(n-1)}(z_0)}{(n-1)!}}_{a_{-1}} \frac{1}{z-z_0}$$

$\square$

③ f has a simple pole at $z_0 \Rightarrow \mathrm{Res}_{z_0}(f) = \lim_{z \to z_0} (z-z_0) f(z)$, compute by L'Hôpital.

Remark: f has a simple pole at z_0 $\overset{\text{def.}}{\Leftrightarrow}$ Laurent series starts at $n=-1$, and $a_{-1} \neq 0$.

(i.e. $\mathrm{ord}_{z_0}(f) = -1$) //

Ex// $\mathrm{Res}_i\left(\frac{e^{\pi z}}{z^2+1}\right) = \lim_{z \to i} \frac{(z-i)e^{\pi z}}{(z^2+1)} = \lim_{z \to i} \frac{e^{\pi z}}{z+i}$

$$= e^{\pi i}/2i = i/2,$$

and $\mathrm{Res}_{-i}\left(\frac{e^{\pi z}}{z^2+1}\right) = \lim_{z \to -i} \frac{e^{\pi z}}{z-i} = \frac{e^{-\pi i}}{-2i} = -i/2$. //

Proof: $f(z) = a_{-1}(z-z_0)^{-1} + \underbrace{a_0 + \dots}_{\text{hol.}}$

$$(z-z_0) f(z) = a_{-1} + \underbrace{(z-z_0)(\text{holo.})}_{\to 0 \text{ as } z \to z_0.}$$

$\square$

Here is another generalization of ①:

④ f has a simple pole at z_0, g holo. at z_0 $\Longrightarrow$

$$\mathrm{Res}_{z_0}(fg) = g(z_0)\underbrace{\mathrm{Res}_{z_0}(f)}_{\text{(compute by ③ or other method)}}$$

Proof: HW

Ex// see below (§ II). //

⑤ f holomorphic at z_0, $f(z_0)=0$, $f'(z_0)\neq 0$ $\Longrightarrow$

$$\mathrm{Res}_{z_0}\left(\frac{1}{f}\right) = \frac{1}{f'(z_0)}.$$

Ex// $\mathrm{Res}_0\left(\dfrac{1}{e^{3z}-1}\right) = 1/3$

$\frac{d}{dz}(e^{3z}-1)\Big|_{z=0} = 3e^{3z}\Big|_{z=0} = 3$ //

Proof: $f(z) = b_1(z-z_0)(1+h(z))$, $h(z_0)=0$ (h holo)

$\Longrightarrow \dfrac{1}{f(z)} = \dfrac{1}{b_1(z-z_0)}(1-h+h^2-\dots)$

$\Longrightarrow$ "a_{-1}" $= \dfrac{1}{b_1} = \dfrac{1}{f'(z_0)}$. ∎

⑥ $\underbrace{\mathrm{ord}_{z_0}(f) = m}_{\text{(z_0 could be a pole or zero)}} \Longrightarrow \mathrm{Res}_{z_0}(f'/f) = m.$

Ex// $\mathrm{Res}_{\zeta}\left(\dfrac{10z^9}{z^{10}-1}\right) = 1$, since $z^{10}-1$ has a "simple zero" at each ζ.

(any 10^{th} root of 1) //

Proof: $f(z) = a_m(z-z_0)^m(1+h(z))$, $h(z_0)=0$ (h holo.)

$$f'(z) = m a_m (z-z_0)^{m-1}(1+h(z)) + a_m (z-z_0)^m h'(z)$$

$$\frac{f'(z)}{f(z)} = \frac{m}{(z-z_0)} + \underbrace{\frac{h'(z)}{1+h(z)}}_{\text{holo. at } z_0} \Longrightarrow \operatorname{Res}_{z_0}\left(\frac{f'}{f}\right) = m. \quad \square$$

II. Residue formulas

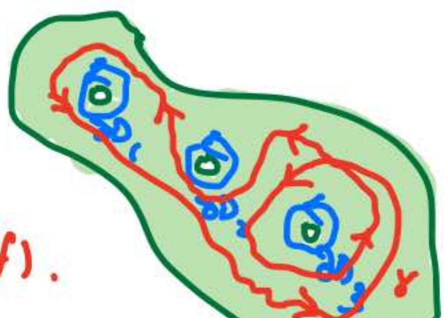
In each case, $U \subseteq \mathbb{C}$ is open, & $\gamma \subset U$ is a closed path which is homologous to zero (in U) and avoids points where we are taking residue.

(a) $f \in \mathrm{Hol}(U \setminus \{z_1, \dots, z_m\}) \Longrightarrow$

("Residue theorem")

$$\boxed{\int_\gamma f\,dz = 2\pi i \sum_{j=1}^{m} W(\gamma, z_j)\operatorname{Res}_{z_j}(f)}$$

Proof: Use
$$\begin{cases} \gamma \underset{\text{hom}}{\equiv} \sum_j W(\gamma, z_j)\,\partial D_j \\ \int_{\partial D_j} f\,dz = 2\pi i \operatorname{Res}_{z_j}(f). \end{cases}$$



$\square$

Ex//
$$\int_{|z|=2} \frac{e^{\pi z}}{1+z^2}\,dz = 2\pi i\left(\operatorname{Res}_i\left(\frac{e^{\pi z}}{1+z^2}\right) + \operatorname{Res}_{-i}\left(\frac{e^{\pi z}}{1+z^2}\right)\right)$$
$$= 2\pi i\left(\frac{i}{2} - \frac{i}{2}\right) = 0. \quad //$$

(see Ex. above)

(b) $f \in \mathcal{M}er(U) \implies$ [writing $\{z_1, \dots, z_m\} = \bigcup \begin{pmatrix} \text{f's zeroes} \\ \text{\& (f's poles)} \end{pmatrix}$]

("argument principle")

$$\boxed{\int_\gamma \frac{f'}{f} dz = 2\pi i \sum_{j=1}^{m} W(\gamma, z_j) \operatorname{ord}_{z_j}(f)}$$

(Proof: use (a) + (b).)

Ex// $\displaystyle\int_{|z|=2} \frac{10z^9}{z^{10}-1} dz = 2\pi i \sum_{j=1}^{10} 1 \cdot 1 = 20\pi i.$ //

(c) $\begin{cases} f \in \mathcal{M}er(U) \\ g \in \mathcal{H}ol(U) \end{cases} \implies$

("generalized argument principle" [GAP])

$$\boxed{\int_\gamma \frac{f'}{f} g \, dz = 2\pi i \sum_j W(\gamma, z_j) \operatorname{ord}_{z_j}(f) g(z_j)}$$

(Proof: use (a), (b), and (4).)

Ex// $\displaystyle\int_{|z|=2} \frac{10z^9 g(z)}{z^{10}-1} dz = 2\pi i \sum_{j=1}^{10} g(\zeta_{10}^j).$ //

These are amusing exercises, but why do we really care about this?

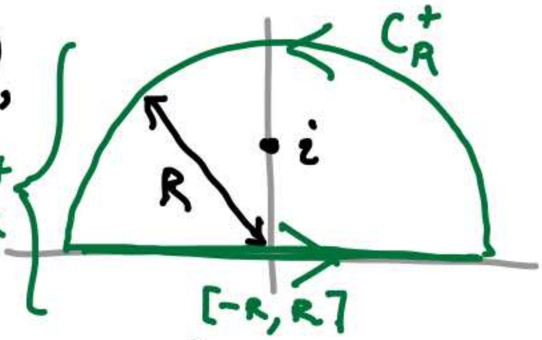
Because we can use it to compute real integrals:

Example Set $I := \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{(1+x^2)^2}\,dx.$

Let $f(z) := \frac{z^2}{(z^2+1)^2} \in \mathcal{H}ol(\mathbb{C}\setminus\{i,-i\})$,

and notice that

$\gamma_R := [-R,R] + C_R^+$



$$\left| \int_{C_R^+} f\,dz \right| \le \pi R \|f\|_{C_R^+} \le \pi R \frac{R^2}{(R^2-1)^2} = \frac{\pi R^3}{R^4 - 2R^2 + 1} \to 0 \text{ as } R\to\infty.$$

for $|z| = R$, $|z^2+1| \ge |z|^2 - 1 = R^2 - 1$.

So $I = \lim_{R\to\infty} \left(\int_{\gamma_R} f\,dz - \int_{C_R^+} f\,dz \right) = \lim_{R\to\infty} \int_{\gamma_R} f\,dz$

$= 2\pi i \,\mathrm{Res}_i(f).$

To evaluate this residue we use ② with $n=2$:

writing $f(z) = \frac{z^2/(z+i)^2}{(z-i)^2} = \frac{F(z)}{(z-i)^2}$, we have

$$F'(z) = \frac{2iz^2 - 2z}{(z+i)^4} \Rightarrow F'(i) = -\frac{i}{4} \Rightarrow \mathrm{Res}_i(f) = \frac{F'(i)}{(2-1)!} = -\frac{i}{4}$$

$\Rightarrow I = 2\pi i\left(-\frac{i}{4}\right) = \frac{\pi}{2}$. //

Slightly more gen'l condition than simple + closed

Definition γ has an interior if $W(\gamma,\alpha) = 0$ or 1 for every $\alpha \in \mathbb{C}\setminus|\gamma|$, and $\mathrm{Int}(\gamma) := \{\alpha \in \mathbb{C} \mid W(\gamma,\alpha) = 1\}$.

Let $f \in \mathrm{Mer}(U)$ & $g \in \mathrm{Hol}(U)$ for some open set U containing both γ and $\mathrm{Int}(\gamma)$. Assume no poles or zeroes of f lie on γ.

Corollary of (b): If γ has an interior, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \sum_{p \in \mathrm{Int}(\gamma)} \mathrm{ord}_p(f) = \begin{pmatrix}\#\text{ of zeroes of } f \\ \text{inside } \gamma, \text{ counted w/mult.}\end{pmatrix} - \begin{pmatrix}\#\text{ of poles of } f \\ \text{inside } \gamma, \text{ counted w/mult.}\end{pmatrix}.$$

Corollary of (c): If γ has an interior, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,g\,dz = \sum_{p \in \mathrm{Int}(\gamma)} \mathrm{ord}_p(f)\,g(p).$$

[Note that Cauchy's integral formula is a special case, with $f(z) = z - a$.]

III. Residues of differentials

It sometimes makes more sense to write "$d\log f$" instead of $\frac{f'(z)}{f(z)}dz$. (Recall that this is not necessarily d of some function $\log f$.†) We have the obvious properties

- $\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g} \Rightarrow d\log(fg) = d\log(f) + d\log(g)$.
- $d\log(\alpha f) = d\log(f)$
- $d\log(f/g) = d\log(f) - d\log(g)$.

† this is locally true, but need not be globally true on U.

This goes beyond a mnemonic for logarithmic differentiation rules: first, for the above "Corollary of (b)", it expresses the fact that you are recording how much $\log(f)$ (or $\arg(f)$) changes around γ. But it also is because of the following

FACT: Residues of $\begin{cases} \text{functions ARE NOT} \\ \text{differentials ARE} \end{cases}$ invariant under local analytic isomorphism.

Here $\mathrm{Res}_{z_0}(F(z)\,dz) := \mathrm{Res}_{z_0}(F(z))$, so they appear to be exactly the same.

(= z)

But now substitute $z = z(w) = z_0 + \underset{\substack{\#\\ 0}}{b_1}(w-w_0)(1+h(w))$ $\quad h(w_0)=0$

and take Res_{w_0}:

$$F(z) \rightsquigarrow F(z(w)) \qquad \text{vs.} \qquad F(z)\,dz \rightsquigarrow F(z(w))\,z'(w)\,dw$$

$$\mathrm{Res}_{w_0}(F(z(w))) \qquad\qquad \mathrm{Res}_{w_0}(F(z(w))\,z'(w)\,dw) = \mathrm{Res}_{w_0}(F(z(w))\,z'(w)).$$

Obviously, these can't be the same in general.

Example // $F(z) = \underbrace{a_{-1}}(z-z_0)^{-1}$, $z(w) = z_0 + b_1(w-w_0)$, $z'(w) = b_1$

$F(z(w)) = a_{-1}b_1^{-1}(w-w_0)^{-1}$, but

$F(z(w))\,z'(w) = a_{-1}\cancel{b_1^{-1}}(w-w_0)^{-1}\cancel{b_1} = \underbrace{a_{-1}(w-w_0)^{-1}}$ has the "right" residue. //

This makes sense, if we write $\mathrm{Res}_{w_0}\big(F(z(w))\,\underbrace{z'(w)\big)}_{dw}$ =

$$\frac{1}{2\pi i}\int_\gamma F(z(w))\underbrace{z'(w)\,dw}_{dz} \underset{\substack{\uparrow \\ (\Delta \text{ of variable})}}{=} \frac{1}{2\pi i}\int_{z\circ\gamma} F(z)\,dz = \mathrm{Res}_{z_0}\big(F(z)\underbrace{\big)}_{dz}$$

where

$z(\gamma)$ or "$z\circ\gamma$". So the differential works because it transforms like the differential in the integral under Δ of variable.

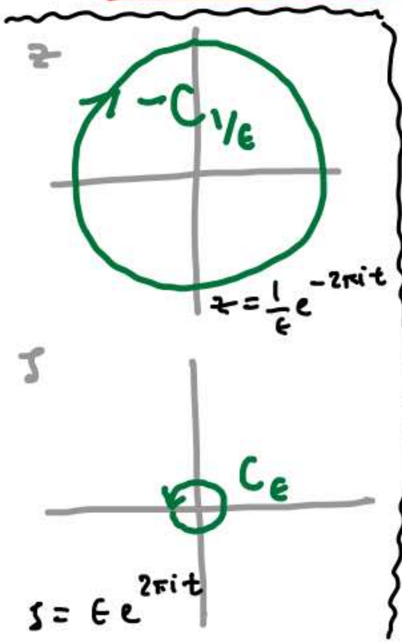
Residue at ∞: Here we transform the differential into the local coordinate at ∞, $\zeta = \frac{1}{z}$ $\left(\longleftrightarrow z = \frac{1}{\zeta}\right)$:

$$f(z)\,dz = f\left(\tfrac{1}{\zeta}\right) d\left(\tfrac{1}{\zeta}\right) = -f\left(\tfrac{1}{\zeta}\right)\frac{d\zeta}{\zeta^2}.$$ (say $f \in \mathrm{Mer}(\mathbb{C})$)

Now assume f has finitely many poles on $\mathbb{C}$, so that $\exists\ \epsilon > 0$ s.t. the only pole enclosed by C_ϵ (in ζ) is at $\zeta = 0$ (if there even is one). Then

$$\mathrm{Res}_\infty(f(z)\,dz) := \mathrm{Res}_0\left(-f\left(\tfrac{1}{\zeta}\right)\frac{d\zeta}{\zeta^2}\right) \left(= \mathrm{Res}_0\left(\frac{-f(1/\zeta)}{\zeta^2}\right)\right)$$

by above defn. of Res (differential)



$$= \frac{1}{2\pi i}\int_{C_\epsilon} \underbrace{\frac{-f\left(\frac{1}{\zeta}\right)}{\zeta^2}\,d\zeta}_{\text{"}dz\text{"}} = \frac{1}{2\pi i}\int_{-C_{1/\epsilon}} f(z)\,dz$$

$$= \frac{-1}{2\pi i}\int_{C_R} f(z)\,dz \quad \text{where } R \gg 0$$

(sufficiently large to enclose ALL poles of f on $\mathbb{C}$).

But then

$$\sum_{p\in S^2} \mathrm{Res}_p(f(z)\,dz) = \sum_{p\in \mathbb{C}} \mathrm{Res}_p(f(z)) + \mathrm{Res}_\infty(f(z)\,dz)$$

$$= \frac{1}{2\pi i}\int_{C_R} f(z)\,dz + \left(\frac{-1}{2\pi i}\int_{C_R} f(z)\,dz\right)$$

$$= 0,$$

and we conclude the

$\boxed{\text{Theorem}}$ The sum of the residues of a differential on S^2 is 0.