Lecture 24: Isolated singularities

I. 3 types of singularities

Let $f \in \mathcal{Hol}(U)$, $U \subset \mathbb{C}$ open. A singularity of f is really just a point (of U^c) where f is not defined.

Definition $p \in U^c$ is an isolated singularity of f if $\exists\, R > 0$ s.t. $D^*(p, R) \subset U$.

(That is, p is not an accumulation point of the other singularities of f.)

Consider the case $p = 0$: let $f \in \mathcal{Hol}(D_R^*)$; then $f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ there, for some unique $\{a_n\}_{n \in \mathbb{Z}}$.

The 3 possibilities for "species" of f's isolated singularity at 0 are:

(i) all $a_n = 0$ for $n < 0$ $\overset{\text{def.}}{\iff}$ 0 "removable" singularity

$\overset{\text{(obvious)}}{\iff}$ can extend f to $\mathrm{Hol}(D_R)$

(ii) all $a_n = 0$ for $n < -m$ $\overset{\text{def.}}{\iff}$ 0 pole of order m
(but $a_{-m} \neq 0$) ($\mathrm{ord}_0(f) = -m$)

(iii) "otherwise" (i.e. $\exists$ ∞'ly many $n < 0$ s.t. $a_n \neq 0$) $\overset{\text{def.}}{\iff}$ 0 essential singularity.

$\boxed{\text{Proposition (Riemann)}}$ f bounded on D_R^* $\overset{\text{obvious}}{\Leftarrow}\!\!\Rightarrow$ 0 removable.

Proof: By the formula for Laurent coefficients,

$$|a_{-(n+1)}| = \left| \frac{1}{2\pi i} \int_{C_\epsilon} w^n f(w)\, dw \right| \leq \frac{2\pi\epsilon}{2\pi} \epsilon^n \|f\|_{C_\epsilon}$$

$$\leq \epsilon^{n+1} B \longrightarrow 0 \quad (\epsilon \to 0)$$

$\|f\|_{D_R^*} =: B$

for all $n \geq 0$. $\square$

A different approach to Riemann's theorem (and a slightly stronger statement) is given by the

$\boxed{\text{Proposition}}$ $\lim_{z\to 0} z f(z) = 0 \overset{\text{obvious}}{\Longleftrightarrow} 0$ removable.

Proof: Using the "can extend f to $\mathcal{H}ol(D_R)$" form of the def'n. of removable singularity, we can forego Laurent series entirely: for $z \in D_R^*$, taking $0 < \epsilon < |z| < r < R \implies$

$$f(z) = \underbrace{\frac{1}{2\pi i}\oint_{C_r} \frac{f(w)}{w-z}\,dw}_{\text{yields holo. fcn. inside } D_R} - \cancel{\frac{1}{2\pi i}\oint_{C_\epsilon} \frac{f(w)}{w-z}\,dw} \to 0$$

("strong" form of Cauchy Thm. from §I of Lect.15)

, hence extending f across the origin. $\square$

By means of Laurent series, one readily verifies the following results on poles:

- $\mathrm{ord}_0(f) = -m \iff \underset{\neq 0 \text{ at } 0}{z^m f(z)} \in \mathcal{H}ol(D_R)$
- $\mathrm{ord}_0(f) + \mathrm{ord}_0(g) = \mathrm{ord}_0(fg)$,

  $\Downarrow$ $\mathrm{ord}_0(f) - \mathrm{ord}_0(g) = \mathrm{ord}_0(f/g)$
- If $f \not\equiv 0$ is holo. with $f(0)=0$, then $\frac{1}{f}$ has a pole at 0.

A much more interesting result is the

$\boxed{\text{Theorem (Casorati–Weierstraß)}}$ 0 essential $\Longrightarrow$

(*) $\boxed{f(D_\epsilon^*) \text{ is dense in } \mathbb{C} \ (\forall \epsilon > 0).}$

Remark: This means that $\lim_{z \to z_0} |f(z)|$ is undefined, even if we think of ∞ as a limit. If the limit is ∞, then you've got a pole, not an essential singularity. //

Proof (by contrapositive): If (*) is NOT the case, then there exists $\alpha \in \mathbb{C}$ s.t. $f(D_\epsilon^*)$ omits a nbhd. of α:

$$|f(z) - \alpha| > \delta \ (>0) \quad \forall\ 0 < |z| < \epsilon$$

$$\Longrightarrow \left| \frac{1}{f(z)-\alpha} \right| < \frac{1}{\delta} \ (<\infty) \quad \forall\ z \in D_\epsilon^*$$

$$\underset{\text{Riemann}}{\Longrightarrow} \frac{1}{f(z)-\alpha} \in \mathcal{H}ol(D_\epsilon)$$

$\Longrightarrow$ $f(z) - \alpha$ has (at worst) a pole at 0

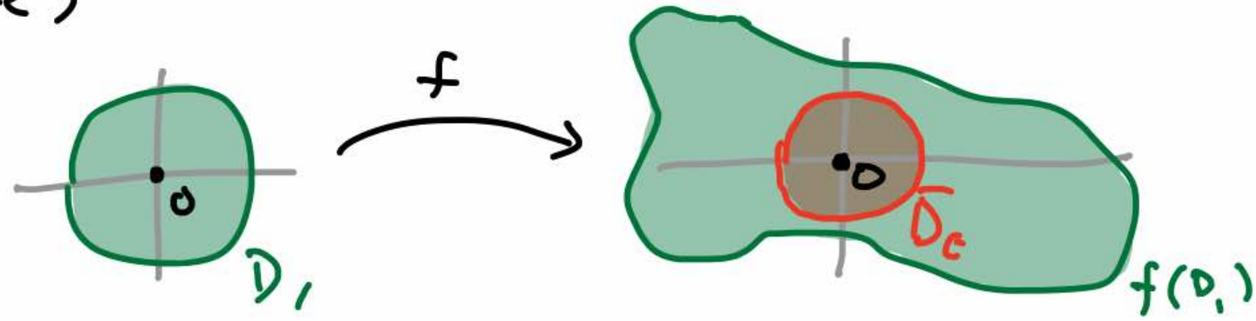
$\Longrightarrow$ $f(z)$ has (at worst) a pole at 0

$\Longrightarrow$ no essential singularity. $\square$

$\boxed{\text{Corollary}}$ The only analytic automorphisms $f:\mathbb{C}\to\mathbb{C}$ are of the form $f(z)=az+b$ (translation-dilation-rotation).

Proof: Assume $f: 0 \mapsto 0$, show $f(z)=az$.

f takes D_1 to an open nbhd. of 0 (since invertible):



So

$$\begin{tikzcd} {|z|>1} \arrow[r, Rightarrow, "f \text{ invertible}"'] & f(z)\notin f(D_1) \arrow[r, Rightarrow] & {|f(z)|>\epsilon} \arrow[d, Rightarrow] \\ {0<|\zeta|<1} \arrow[u, Rightarrow, "\zeta=\frac{1}{z}"] & & {\left|f(\tfrac{1}{\zeta})\right|>\epsilon} \end{tikzcd}$$

Set $h(\zeta) := f\left(\frac{1}{\zeta}\right)$: then $h(D_1^*)$ omits a nbhd. of 0

$(|h(\zeta)|>\epsilon \quad \forall \zeta\in D_1^*)$

$\underset{C-W}{\Longrightarrow}$ h has at worst a pole at 0.

But now $f(z) = \sum_{n\geq 0} a_n z^n \Rightarrow h(\zeta) = f(\frac{1}{\zeta}) = \sum_{n\geq 0} a_n \zeta^{-n}$

must terminate for $n>m$, so must have only $a_0, a_1, \ldots, a_m$ possibly nonzero.

That is, $f(z) =$ polynomial $= a\prod(z-z_i)$.

fund. thm. algebra

If z_i not all same, then $f(z_i) = f(z_j) = 0 \implies$

f not injective $\implies$ f not automorphism.

Thus $f(z) = a(z-z_0)^m \implies m = 1$, $f(z) = a(z-z_0)$.

again use injectivity of f

But if $f(0) = 0$, then $z_0 = 0$; done. $\square$

Remark: The Big Picard Theorem is a substantial strengthening of C-W which we will prove later in this course. It says that:

If f has an essential singularity at 0, then on any D_ϵ^*, f takes on all possible complex values, with at most a single exception, infinitely often.

Example // $e^{1/z}$ has an essential singularity at 0, and obviously omits the value 0. //

We conclude this part with an application of Riemann's theorem:

$\boxed{\text{Taylor's theorem}}$ $f \in \mathrm{Hol}(U)$, $a \in U \implies$

$$f(z) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(z-a)^k + f_n(z)(z-a)^n$$

where $f_n \in \mathrm{Hol}(U)$, $f^{(n)}(a) = n!\, f_n(a)$, and

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial D(a,r)} \frac{f(w)}{(w-a)^n (w-z)}\, dw \quad \text{for } |z-a| < r.$$

$\cap U$

Proof: Since $g_1(z) := \dfrac{f(z)-f(a)}{z-a}$ times $(z-a)$ goes to 0 as $z \to a$, Riemann $\implies$ $g_1 = f_1|_{U \setminus \{a\}}$ for $f_1 \in \mathrm{Hol}(U)$.

Repeat with $g_2(z) = \dfrac{f_1(z) - f_1(a)}{z-a} \underset{\substack{\text{extend} \\ \text{(by Riemann)}}}{\rightsquigarrow} f_2 \in \mathrm{Hol}(U)$, etc.

This gives
$$\begin{aligned} f(z) &= f(a) + (z-a) f_1(z) \\ &= f(a) + (z-a) f_1(a) + (z-a)^2 f_2(z) \\ &= \cdots, \end{aligned}$$

hence
$$\begin{aligned} 2\pi i\, f_n(z) &= \int_{\partial D} \frac{f_n(w)}{w-z}\, dw \\ &= \int_{\partial D} \frac{f(w)\, dw}{(w-a)^n (w-z)} - \left\{ \begin{array}{l} \text{terms of the form} \\ \text{Const.} \times \displaystyle\int_{\partial D} \frac{dw}{(w-a)^\ell (w-z)} \end{array} \right\} \end{aligned}$$

$\displaystyle\int_{\partial D} \frac{dw}{(w-a)^\ell (w-z)} = F_\ell(a)$, $0 < \ell \le n$.

variable constant

But $(z-a)F_1(a) = W(\partial D, a) - W(\partial D, z) = 0$,

and $F_\ell(a) = \text{const.} \times F_1^{(\ell-1)}(a) = 0$. $\square$

Remark: Ahlfors uses Taylor's Thm. to

(i) bound the error in the Taylor series partial sums:

$$|f_n(z)(z-a)^n| \le \frac{1}{\cancel{2\pi}}\cancel{2\pi} r \frac{\|f\|_{\partial D}\,|z-a|^n}{r^n(r-|z-a|)} \xrightarrow[(n\to\infty)]{} 0 \quad \text{for } \frac{|z-a|}{r} \le 1-\epsilon$$

$\Rightarrow$ uniform convergence on compact sets

(ii) show that zeroes are isolated: a similar estimate demonstrates that if all $f^{(k)}(a)$ vanish then f vanishes on an open disk $\Rightarrow$ the set S where all $f^{(k)}$ vanish is open. Obviously S^c is open. So U = region $\Rightarrow S$ or $S^c = \emptyset$. If $f \not\equiv 0$, then $U \neq S$ so $U = S^c$; locally at a zero, some $f^{(k)}$ is $\neq 0$ $\Rightarrow$ in a punctured nbhd. $f \neq 0$. //

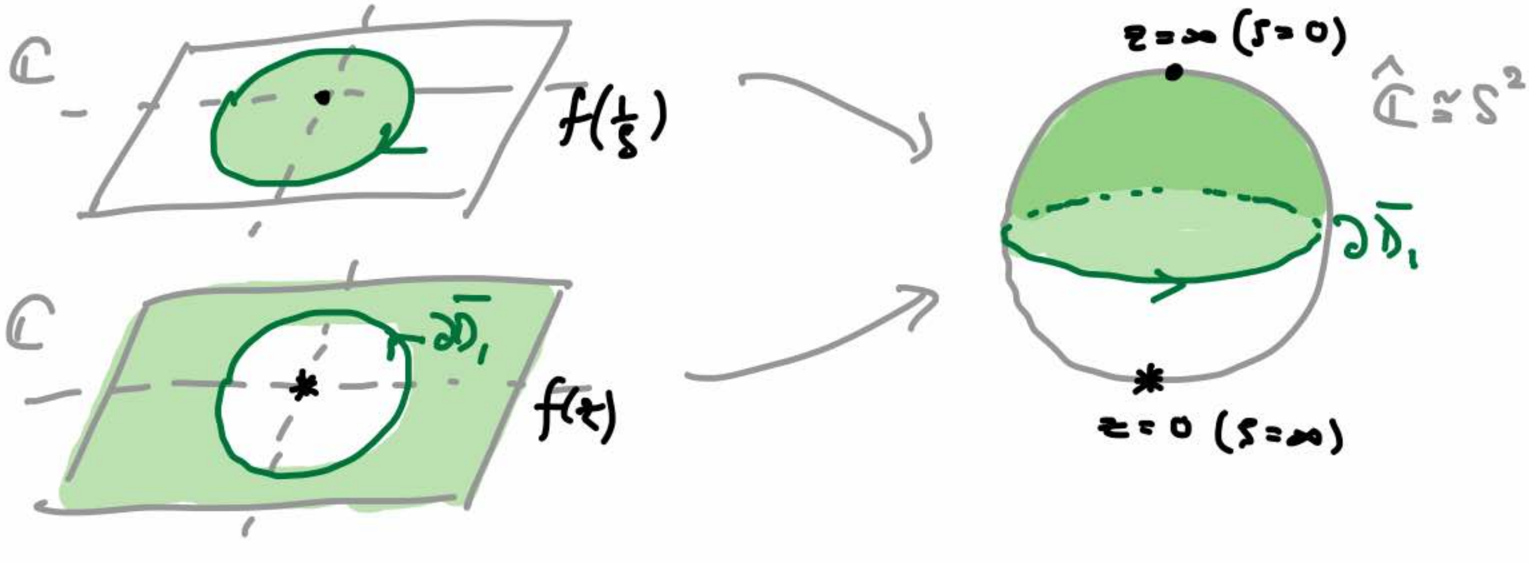
II. Meromorphic functions

We write (for $U \subset \mathbb{C}$ open)

$f \in \mathcal{M}er(U) \overset{\text{def.}}{\Longleftrightarrow}$ f restricts to a holo. fcn. on the complement of a discrete set $S \subset U$ (no limit pts. in U), with poles at each point of S.

Example// If $g_1, g_2 \in \mathcal{H}ol(U) \setminus \{0\}$ ← the "zero function", we have $g_1/g_2 \in \mathcal{M}er(U)$. //

The Riemann sphere: Given $f(z)$ on $\mathbb{C}$ (holo/mero/whatever), consider $f\left(\frac{1}{s}\right)$, $s = \frac{1}{z} \leftrightarrow z = \frac{1}{s}$. Notice that f is completely described by $f(z)$ for $z \in \overline{D}_1$ and $f\left(\frac{1}{s}\right)$ for $s \in \overline{D}_1$:



We say $f(z)$ has a singularity of a certain type at $(z=)\infty \iff f(\frac{1}{\zeta})$ has this type of singularity at $(\zeta=)\,0$. (Same goes for zero/pole of certain order at ∞.)

Now given a cover of S^2 by open sets $\left(= \text{open set in } \mathbb{C}, \text{ or } \{\infty\} \cup \{\text{complement of a compact set in } \mathbb{C}\}\right)$, we get finite subcovers on each of $\underbrace{\overline{D}_{1,z}}_{\text{lower hemisphere}}$ & $\underbrace{\overline{D}_{1,\zeta}}_{\text{upper hemisphere}}$

$\Longrightarrow S^2$ compact.

<u>Remark</u>: ∞ is an accumulation point of $\mathbb{Z}$.

So for example the zeros of $\cos(z)$ have ∞ as an accumulation point; this guarantees that it has ∞ as an essential singularity. //

$\boxed{\text{Proposition}}$ $f \in \mathrm{Mer}(\mathbb{C})\setminus\{0\} \Rightarrow f$ has finitely many zeros and poles in $\overline{D}_1$ (& more generally any compact set).

Proof: If $\{\text{polar set of } f\} \cap \overline{D}_{1+\epsilon}$ is infinite, then it has an accumulation point $\Rightarrow$ not discrete $\Longrightarrow f$ not mero.

So this set is finite: the poles of f in $\overline{D}_{1+\epsilon}$ are

$\{\alpha_1, \dots, \alpha_m\}$. Let $U = D_{1+\epsilon} \setminus \{\alpha_1, \dots, \alpha_m\}$.

If f has ∞ many zeroes in $\overline{D_1}$, then they have an accumulation point in $\overline{D_1}$ which cannot be one of the $\{\alpha_i\}$. (Near α_i, $|f|$ is big.) Hence, the zeroes have accumulation point in $U \Rightarrow f \equiv 0$, contradiction. $\square$

Corollary $f \in Mer(S^2) \Rightarrow f$ has finitely many zeroes and poles.

Proof: Writing $\zeta = \frac{1}{z}$, $S^2 = \{|z| \le 1\} \cup \{|\zeta| \le 1\}$ and $f(z) \in Mer(\mathbb{C})$, $f(\frac{1}{\zeta}) \in Mer(\mathbb{C})$ (by restriction). Now apply the Proposition. $\square$

Theorem $f \in Mer(S^2) \Longrightarrow f$ rational (quotient of polynomials).

Proof: $\text{ord}_{\infty}(f(z)) = \text{ord}_0\left(f(\tfrac{1}{\zeta})\right) = m$, and

$\text{ord}_{\alpha_i}(f) = m_i \in \mathbb{Z} \setminus \{0\}$ for finitely many $\alpha_i \in \mathbb{C}$.

Hence $G(z) := \dfrac{f(z)}{\prod (z - \alpha_i)^{m_i}}$ has NO zeroes or poles on $\mathbb{C}$

$\Rightarrow G, \frac{1}{G}$ entire. At ∞, $\mathrm{ord}_\infty(G) = m + \sum m_i$

$$\Longrightarrow \lim_{z\to\infty} |G(z)| = \begin{cases} \infty \Rightarrow 1/G \text{ bounded} \\ \left.\begin{matrix}\text{finite} \\ 0\end{matrix}\right\} \overset{S^2 \text{ compact}}{\Longrightarrow} G \text{ bounded} \end{cases} \overset{\text{Liouville}}{\Longrightarrow} G \ \underline{\text{const}}.$$

So $C = \dfrac{f(z)}{\prod (z-\alpha_i)^{m_i}} \Rightarrow f(z) = C \prod (z-\alpha_i)^{m_i} = \dfrac{P(z)}{Q(z)}$,

where $P(z) = C \prod\limits_{i: m_i > 0} (z-\alpha_i)$, $Q(z) = \prod\limits_{i: m_i<0} (z-\alpha_i)^{-m_i}$. $\square$

Corollary $f \in \mathrm{Mer}(\mathbb{C})$, $\lim\limits_{|z|\to\infty} |f(z)| = \infty \Rightarrow f$ rational

Proof: For $|z| > \frac{1}{\delta}$, $|f(z)| > \frac{1}{\epsilon}$; i.e.

$|\xi| < \delta$ gives $|f(\frac{1}{\xi})| > \frac{1}{\epsilon}$. Now Casorati–Weierstraß $\Longrightarrow f(\frac{1}{\xi})$ doesn't have an essential singularity $\Rightarrow$ it has a pole. Apply last Theorem. $\square$

Corollary (a) $f \in \mathrm{Hol}(\mathbb{C})$ with pole at $\infty \Rightarrow$ polynomial

(b) $f \in \mathrm{Hol}(S^2) \Rightarrow f$ constant.

Proof: (a) $f = \frac{P}{Q}$ by Theorem; Q can't have zero on $\mathbb{C}$ $\Rightarrow Q$ const.

(b) by (a), $f = P = C\prod(z-\alpha_i)^{k_i}$ $(k_i \geq 0)$

$\Rightarrow \mathrm{ord}_\infty(f) = -\sum k_i$. But $\mathrm{ord}_\infty(f) \geq 0 \Rightarrow$ all $k_i = 0$. $\square$

It will turn out that all simply connected Riemann surfaces (complex 1-manifolds) are bi-holomorphic to S^2, D_1, or $\mathbb{C}$. If we know the automorphism groups of the latter, we know them for the RS. With this in mind:

- $\mathrm{Aut}(D_1) \cong PSL_2(\mathbb{R})$
- $\mathrm{Aut}(\mathbb{C}) \cong \mathbb{R}^* \ltimes \mathbb{R}$ (via transformations $z \mapsto az+b$)
- $\mathrm{Aut}(S^2) \cong PSL_2(\mathbb{C})$.

Proof: Since $FLT \cong PSL_2(\mathbb{C})$, suff. to show $\mathrm{Aut}(S^2) = FLT$.

The $\supseteq$ inclusion is clear. Now let $f \in \mathrm{Aut}(S^2)$, i.e. f is a 1-1 holo. map $S^2 \to S^2$, hence a meromorphic function with a single, simple, pole. By the Corollary above, f is a rational function P/Q. Written in lowest terms, Q must have degree ≤ 1. In order to be 1-1, also, $\deg(P) \leq 1$. So f is a FLT. $\square$