Lecture 23: Function series

I. Uniform limits of analytic functions

Let $U \subset \mathbb{C}$ be an open set.

Proposition Let $\{f_n : U \to \mathbb{C}\}$ converge uniformly on compact subsets of U, and write

$$f(z) := \lim_{n\to\infty} f_n(z)$$

for the pointwise limit. Then

$$f_n \in \begin{cases} C^0(U) \\ \text{resp.} \\ \mathcal{H}ol(U) \end{cases} (\forall n) \implies f \in \begin{cases} C^0(U) \\ \text{resp.} \\ \mathcal{H}ol(U) \end{cases}.$$

In the "$\mathcal{H}ol$" case, the $\{f_n'(z)\}$ also converge uniformly on compact sets, to $f'(z)$.

Proof: The $C^0(U)$ case is an earlier result, so assume $f_n \in \mathcal{H}ol(U)$ $(\forall n)$. Given $z_0 \in U$, and $0 < R < d(z_0, U^c)$, $\{f_n\}$ converges uniformly on $\overline{D}(z_0, R)$.

For $z \in \overline{D}(z_0, R/2)$,

$$f_n(z) = \frac{1}{2\pi i}\int_{\partial\overline{D}(z_0,R)} \frac{f_n(w)}{w-z}\,dw$$

by the local Cauchy formula (since f_n is holomorphic).

So for $z \in \overline{D}(z_0, R/2)$,

$$\left| f_n(z) - \frac{1}{2\pi i}\int_{\partial\overline{D}(z_0,R)} \frac{f(w)\,dw}{w-z} \right| = \frac{1}{2\pi}\left| \int_{\partial\overline{D}(z_0,R)} \frac{f_n(w)-f(w)}{w-z}\,dw \right|$$

$$\leq \frac{1}{\cancel{2\pi}}\cancel{2\pi} R \left\| \frac{f_n(w)-f(w)}{w-z} \right\|_{(w,z)\in\partial\overline{D}(z_0,R)\times\overline{D}(z_0,R/2)}$$

$$\leq \frac{\cancel{R}}{\cancel{R}/2}\|f_n - f\|_{\partial\overline{D}(z_0,R)}$$

$$\xrightarrow[(n\to\infty)]{} 0$$

by uniform convergence on $\overline{D}(z_0, R)$.

by defn., the function to which $\{f_n\}$ converges

Hence $f(z) = \frac{1}{2\pi i}\int_{\partial D(z_0,R)} \frac{f(w)}{w-z}\,dw$ for $z \in \overline{D}(z_0, R/2)$.

By the result on functions given by integrals over paths, f <u>must</u> be holomorphic there, with

$$f'(z) = \frac{1}{2\pi i}\int_{\partial D(z_0,R)} \frac{f(w)}{(w-z)^2}\,dw.$$

Since $f_n'(z) = \frac{1}{2\pi i}\int_{\partial D(z_0,R)} \frac{f_n(w)\,dw}{(w-z)^2}$,

$$\|f_n'(z) - f'(z)\|_{\bar{D}(z_0,R/2)} = \frac{1}{2\pi}\left\| \int_{\partial D(z_0,R)} \frac{f_n(w)-f(w)}{(w-z)^2}\,dw \right\|_{(z\in)\,\bar{D}(z_0,\frac{R}{2})}$$

$$\leq R\left\| \frac{f_n(w)-f(w)}{(w-z)^2}\right\|_{\partial\bar{D}(z_0,R)\times\bar{D}(z_0,R/2)}$$

$$\leq \frac{R}{R^2/4}\|f_n(w)-f(w)\|_{\partial\bar{D}(z_0,R)}$$

$$\xrightarrow[(n\to\infty)]{} 0.$$

Finally, any compact set can be covered by a finite union of balls $D(z_i,R_i)$ with $R_i < d(z_i,U^c)$, hence by the $\bar{D}(z_i,R_i)$ $(\subset U)$. [This is because

$$D(K,U^c) > (\epsilon >)\, 0$$

$\Rightarrow$ we can take all $R_i = \epsilon$, for example.] $\square$

<u>Remark</u>: <u>(i)</u> If $\bigcup_\alpha U_\alpha = U$ (for a collection of open $U_\alpha \subset U$), then any compact $K\subset U$ is contained in a <u>finite</u> union of U_α's. Hence it is enough to show uniform convergence on each U_α in order to

satisfy the condition of the proposition.

(ii) Alternatively we can start with $f_n \in \mathcal{H}ol(U_n)$,

$$U_1 \subset U_2 \subset \cdots \subset U = \bigcup_n U_n .$$

Since any compact $K \subset U$ is contained in some U_N, the hypothesis still makes sense (and the conclusion still holds for the pointwise limit f). //

Example (using (ii)) // $f_n(z) = \frac{z}{2z^n+1} \in \mathcal{H}ol(D_{1/\sqrt[n]{2}})$,

and $D_1 = \bigcup_n D_{1/\sqrt[n]{2}}$. The pointwise limit on D_1 is $\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \frac{z}{2z^n+1} \underset{|z|<1}{=} \frac{z}{1} = z$, and

$$f_n(z) - z = \frac{z - 2z^{n+1} - z}{2z^n+1} = \frac{-2z^{n+1}}{2z^n+1} \to 0 \text{ uniformly}$$

on any $\overline{D_{r<1}}$ (bound by $\frac{2r^{n+1}}{1-2r^n}$), and any compact $K \subset D_1$ is in such a $\overline{D_r}$. So we reach, by the Proposition, the very exciting conclusion that z is holomorphic on D_1. //

Example (using (i)) // A Dirichlet character mod k $k \in \mathbb{N}$

is a function

$$\chi : \mathbb{N} \to \mathbb{C}$$

satisfying

- $\chi(1) = 1$
- $\chi(n) = 0$ if $\gcd(n,k) \neq 1$
- $\chi(n+k) = \chi(n)$
- $\chi(nm) = \chi(n)\chi(m)$

e.g. $(k=4)$ $\underbrace{0\ 1\ 0\ -1}\ \underbrace{0\ 1\ 0\ -1}\ \ldots$

$(k=5)$ $\underbrace{0\ 1\ i\ -i\ -1}\ \underbrace{0\ 1\ i\ -i\ -1}\ \ldots$

The corresponding L-function $L(\chi, z)$ is defined

by a series for $\mathrm{Re}(z) > 1$,

$$f(z) := \sum_{m \geq 1} \underbrace{\frac{\chi(m)}{m^z}}_{F_m(z)} \qquad m^z = e^{z \log m}$$

These functions are immensely important in number theory.

(A similar function is the Riemann zeta function $\sum_{n\geq 1} \frac{1}{n^z} =: \zeta(z)$)

Let

$$f_n(z) := \sum_{m=1}^{n} F_m(z),$$

which is analytic for $\mathrm{Re}(z) > 1$ (in fact, for $z \in \mathbb{C}$).

For $Re(z) \geq c\ (>1)$,

$$|m^{-z}| = |e^{-z\log m}| = |e^{-Re(z)\log m}|\,\cancel{|e^{-iIm(z)\log m}|}^{\,1}$$

$$\leq |e^{-c\log m}| = m^{-c}.$$

Moreover, $\sum_{m\geq 1} \frac{1}{m^c}$ converges (since $c>1$) by the integral test, so the Weierstraß M-test* (cf. Lecture 5 §II)

$\Longrightarrow$ $f_n(z)$ converges uniformly on $U_c := \{z\in\mathbb{C} \mid Re(z)>c\}$.

Now $\bigcup_{c>1} U_c = U_1$, and so the Proposition applies (via (i)) and $f(z)$ is a holomorphic function on U_1. //

One final application is

$\boxed{\text{Hurwitz's Theorem}}$ Let $f_n \in \mathcal{H}ol(U)$, with $f_n \to f$ uniformly on compact subsets. If $(\forall n \geq N)$ f_n is never 0 on U, then either $f\equiv 0$ or f is never 0 on U.

*NOTE: $\frac{1}{m^c} \geq \left|\frac{1}{m^z}\right|$, but $\left|\frac{\chi(m)}{m^z}\right|$? Yes:

it turns out that $\chi(m)$ is always either 0 or a root of 1, of order $= \varphi(k)$, $\varphi =$ Euler phi-function.

Proof: Assume $f \not\equiv 0$. Given any $z_0 \in U$,

$\exists\ \epsilon > 0$ s.t. $f(z) \neq 0$ for $z \in \overline{D}^*(z_0, \epsilon)\ (\subset U)$.

(This is because zeroes of holomorphic functions are isolated.)

Set $\mu := \min\limits_{z \in \partial D(z_0,\epsilon)} |f(z)|$; then for $n \gg 0$

$$\left\| \frac{1}{f_n} - \frac{1}{f} \right\|_{\partial D(z_0,\epsilon)} = \left\| \frac{f - f_n}{f_n f} \right\|_{\partial D} \leq \frac{\| f - f_n \|_{\partial D}}{\frac{1}{2}\mu^2} \xrightarrow[(n\to\infty)]{} 0 .$$

By the Proposition, $\| f_n' - f' \|_{\partial D} \to 0$, so

$$\left\| \frac{f_n'}{f_n} - \frac{f'}{f} \right\|_{\partial D} \to 0 \quad \text{and}$$

$$2\pi i \underbrace{\eta(f, \partial D)}_{\left(\substack{\#\text{ of zeroes of} \\ f \text{ enclosed by } \partial D}\right)} = \oint_{\partial D} \frac{f'}{f}\, dz = \lim_{n\to\infty} \oint \frac{f_n'}{f_n}\, dz = \lim_{n\to\infty} 2\pi i\, \eta(f_n, \partial D)$$

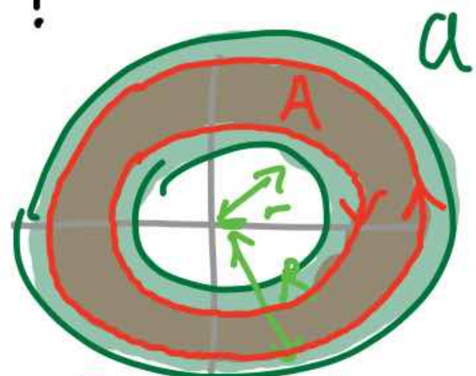
$$= 0 .$$

So f has no zero at z_0. $\square$

II. Laurent series

We know how to write series representing holomorphic functions on a disk. What if they're not holomorphic at the center — or more generally, if $f \in \mathcal{H}ol(\mathcal{A})$ where $\mathcal{A}$ is the annulus

$$\mathcal{A} = \left\{ z \in \mathbb{C} \;\middle|\; r < |z| < R \right\} \; ?$$



$\Gamma := C_S - C_s$

If we pick a radius ϱ between r & R, and let $\gamma = \partial D_\varrho$, then

$$g(z) := \int_\gamma \frac{f(w)}{w-z}\, dw$$

is holomorphic on D_ϱ. Unless f actually had holomorphic extension to D_R, this isn't the right function!

Instead set $s := r+\epsilon$, $S := R-\epsilon$,

$$A := \left\{ z \in \mathbb{C} \;\middle|\; s < |z| < S \right\}.$$

For $z \in A$, the (homology version of) Cauchy integral formula gives for $\Gamma := C_S - C_s \left(= \partial A \underset{\text{hom}}{\equiv} 0 \text{ in } \mathcal{A} \right)$

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z}\, dw \qquad (\text{since } W(\Gamma, z) = 1)$$

$$= \frac{1}{2\pi i}\int_{C_\delta} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{f(w)}{w-z}\,dw$$

$$\underbrace{\left(\text{for } w \in C_\delta,\ \left|\frac{z}{w}\right| < 1\right) \qquad \left(\text{for } w \in C_\varepsilon,\ \left|\frac{w}{z}\right| < 1\right)}_{(*)}$$

$$= \frac{1}{2\pi i}\int_{C_\delta} \sum_{n\geq 0} \frac{f(w)\,z^n}{w^{n+1}}\,dw + \frac{1}{2\pi i}\int_{C_\varepsilon} \sum_{n\geq 0} \frac{f(w)\,w^n}{z^{n+1}}\,dw$$

uniform convergence (by (*))

$$= \sum_{n\geq 0} \underbrace{\left(\frac{1}{2\pi i}\int_{C_\delta} \frac{f(w)}{w^{n+1}}\,dw\right)}_{=:\,a_n} z^n + \sum_{n\geq 0} \underbrace{\left(\frac{1}{2\pi i}\int_{C_\varepsilon} w^n f(w)\,dw\right)}_{=:\,a_{-(n+1)}} z^{-(n+1)}$$

$$= \sum_{m\in\mathbb{Z}} a_m z^m$$

(taking $\varepsilon \to 0$, we see that this is valid on all of $\mathcal{A}$)

$\Longrightarrow$ $\exists$ of Laurent expansions

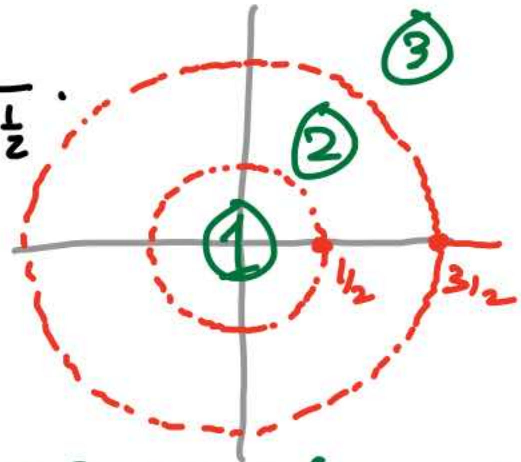
(but one doesn't usually use these formulas to compute the $\{a_n\}$)

<u>Remark</u>: (i) The above computation exhibits $f \in \mathcal{H}ol(\mathcal{A})$ as the sum of $f_1 \in \mathcal{H}ol(D_R)$ and $f_2 \in \mathcal{H}ol(\overline{D_r}^{\,c})$.

(ii) Uniqueness of Laurent expansions: if $\sum_{m\in\mathbb{Z}} a_m z^m$ converges in $\mathcal{A}$ to zero, then the above formulas for the $\{a_n\}$ <u>do</u> recover them, and obviously yield zero $(\forall n)$. //

Example 1

$$f(z) = \frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{3}{2}\right)} = \frac{1}{z-\frac{3}{2}} - \frac{1}{z-\frac{1}{2}}.$$



(3 regions for 3 different Laurent expansions)

① $f(z) = 2\,\frac{1}{1-2z} - \frac{2}{3}\,\frac{1}{1-\frac{2}{3}z}$

② $f(z) = -\frac{1}{z}\,\frac{1}{1-\frac{1}{2z}} - \frac{2}{3}\,\frac{1}{1-\frac{3}{2}z}$

③ $f(z) = -\frac{1}{z}\,\frac{1}{1-\frac{1}{2z}} + \frac{1}{z}\,\frac{1}{1-\frac{3}{2z}}$

e.g. $\left|\frac{1}{2z}\right| < 1 \iff |z| > \frac{1}{2}$

$\left|\frac{3}{2}z\right| < 1 \iff |z| < \frac{3}{2}$

rewrite as shown and expand in geometric series to get Laurent series for $f|_{①}$, $f|_{②}$, & $f|_{③}$. The first is just a power-series expansion.

Example 2

$$\cos\left(\frac{1}{z}\right) = \sum_{m\geq 0} \frac{(-1)^m}{(2m)!}\left(\frac{1}{z}\right)^{2m} \quad \text{on } \mathbb{C}^*,$$

by plugging $\frac{1}{z}$ into the power series for cos.

Example 3

Recall that the Bernoulli #'s are defined by

$$\frac{z}{e^z - 1} = \sum_{n \geq 0} \frac{B_n}{n!} z^n = 1 - \frac{z}{2} + \sum_{m \geq 1} \frac{B_{2m}}{(2m)!} z^{2m}.$$

Now

$$\frac{2iz}{e^{2iz} - 1} = z \cdot \frac{2i}{e^{iz}\left(e^{iz} - e^{-iz}\right)}$$

$$= z \cdot \frac{e^{-iz}}{\sin(z)}$$

$$= z \cdot \frac{\cos z - i \sin z}{\sin z}$$

$$= z \cot z - iz$$

$$\Rightarrow \quad z \cot z = \cancel{iz} + 1 - \cancel{\frac{(2iz)}{2}} + \sum_{m \geq 1} \frac{B_{2m}}{(2m)!} (2iz)^{2m}$$

$$= 1 + \sum_{m \geq 1} \frac{B_{2m} (-4)^m}{(2m)!} z^{2m}$$

$$\Rightarrow \quad \boxed{\cot z = \frac{1}{z} + \sum_{m \geq 1} \frac{B_{2m} \cdot (-4)^m}{(2m)!} z^{2m-1}, \quad \text{valid on } 0 < |z| < \pi.}$$