# Lecture 22: The Poincaré metric

In this (short) lecture, we give another (more intuitive) view of Schwarz–Pick by introducing a bit of differential geometry.

Let $U \subseteq \mathbb{C}$ be a region. A conformal metric on $U$ is a non-negative function $\rho \in C^\infty(U)$, or more precisely the expression $\rho(z)\,|dz|$.

For $z \in U$, $\xi \in \mathbb{C}$ (think of this as a vector), the length of $\xi$ at $z$ (with respect to $\rho$) is

$$\|\xi\|_{\rho,z} := \rho(z) \cdot |\xi|.$$

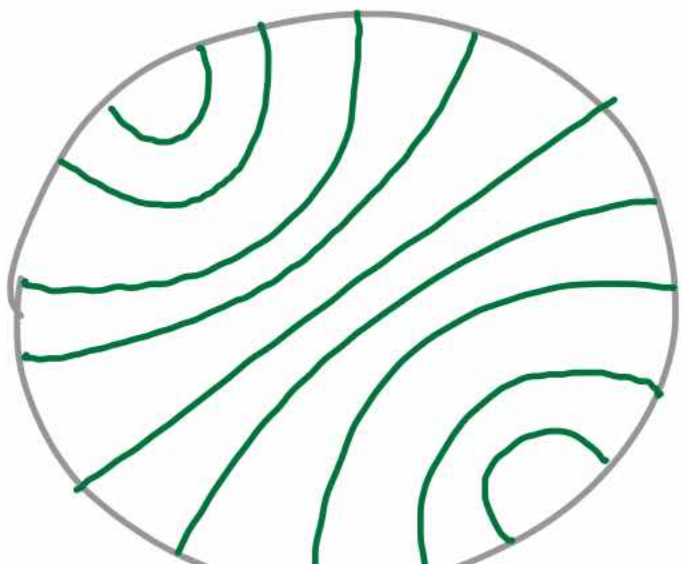
If $\gamma : [a,b] \to U$ is a $C^1$ path, the length of $\gamma$ (with resp. to $\rho$) is

$$L_\rho(\gamma) := \int_a^b \|\gamma'(t)\|_{\rho,\gamma(t)}\, dt.$$

**Example** Let $U = \Delta$. The Poincaré metric is defined by

$$\rho(z) = \frac{1}{1 - z\bar{z}}.$$

- $(\Delta, \rho)$ is a complete metric space (with the distance between two points given by geodesic distance $= L_\rho$ of the "shortest" path) called the hyperbolic disk.
- geodesics are (generalised) circles intersecting $|z| = 1$ at right angles.
- historically the first example of a non-Euclidean geometry (Gauss, Bolyai, Lobachevsky).



Given two regions equipped with metrics

$$(U, \rho), \quad (V, \psi),$$

a holomorphic map $f : \underset{z}{U} \to \underset{w}{V}$ is <u>isometric</u>

$$\overset{\text{def.}}{\iff} \quad f^*\big(\psi(w)|dw|\big) = \rho(z)\,|dz|$$

$$\begin{array}{c} \| \\ (\psi \circ f)(z) \cdot \left|\frac{\partial f}{\partial z}(z)\right| |dz| . \end{array}$$

$\boxed{\text{Theorem}}$ Let $f : \Delta \to \Delta$ be holomorphic, and $\rho$ = Poincaré metric on $\Delta$. Then:

(i) $f$ is distance decreasing (with respect to $\rho$), i.e. $(\rho \circ f) \cdot |f'| \leq \rho$ ;

and

(ii) $f$ is an isometry (i.e. $(\rho \circ f) \cdot |f'| = \rho$ everywhere) $\iff$ $f \in \mathrm{Aut}(\Delta)$.

**Proof:** (i) We want to show

$$(\rho\circ f)\,|f'| \left(= \frac{|f'(z)|}{1-|f(z)|^2}\right) \le \frac{1}{1-|z|^2}.$$

But Schwarz - Pick is

$$|f'(z)| \le \frac{1-|f(z)|^2}{1-|z|^2}.$$

(ii) ($\Longrightarrow$): "$=$" means equality in Schwarz-Pick, done.

($\Longleftarrow$): Since every holomorphic automorphism of $\Delta$ can be written as a composition of $\mu_{e^{i\theta}}$'s & $\phi_\alpha$'s, it suffices to check that:

- $\underbrace{\mu_{e^{i\theta}} \text{ is an isometry}}$:

  $\rho\circ\mu_{e^{i\theta}} = \rho$, and $|\mu'_{e^{i\theta}}(z)| = |e^{i\theta}| = 1$.

- $\underbrace{\phi_\alpha \text{ is an isometry}}$:

$$(\rho\circ\phi_\alpha)(z)\cdot|\underbrace{\phi'_\alpha(z)}_{\frac{1-\alpha\bar\alpha}{(1-\bar\alpha z)^2}}| = \frac{1}{1-\left|\frac{z-\alpha}{1-\bar\alpha z}\right|^2}\cdot\frac{1-|\alpha|^2}{|1-\bar\alpha z|^2}$$

$$= \frac{1-|\alpha|^2}{(1-\bar{\alpha}z)(1-\alpha\bar{z}) - (z-\alpha)(\bar{z}-\bar{\alpha})}$$

$$= \frac{1-|\alpha|^2}{1-\bar{\alpha}z\cancel{-\alpha\bar{z}}+|\alpha z|^2-|z|^2+\cancel{\alpha\bar{z}+\bar{\alpha}z}-|\alpha|^2}$$

$$= \frac{\cancel{1-|\alpha|^2}}{\cancel{(1-|\alpha|^2)}(1-|z|^2)}$$

$$= \frac{1}{1-|z|^2} = \rho(z). \qquad \square$$

If you like differential geometry, you're probably already sold on this. But if not, why should we care about this reformulation, beyond the fact that it simplifies the expression of Schwarz-Pick and makes it more conceptual?

$\boxed{\text{Corollary}}$ If the closure $\overline{f(\Delta)}$ belongs to $\Delta$, then $f$ has a unique fixed point $P$, and this $P$ is the intersection of the sets $V_i = \underbrace{(f\circ\cdots\circ f)}_{i \text{ times}}(K)$ for any sufficiently large compact $K \subset \Delta$.

**Proof (Sketch):** Let $\epsilon = \frac{1}{2} d(\overline{f(\Delta)}, \Delta^c)$,

and fix $z_0 \in \Delta$; then

$$g(z) := f(z) + \epsilon(f(z) - f(z_0)) \quad \text{maps } \Delta \to \Delta$$

$$\underset{\text{Theorem}}{\Longrightarrow} (\rho \circ g)(z) \cdot \underbrace{|g'(z)|}_{(1+\epsilon) f'(z)} \leq \rho(z)$$

$$\Longrightarrow (\rho \circ g)(z_0) \cdot |g'(z_0)| \leq \rho(z_0)$$

$$\Longrightarrow (\rho \circ f)(z_0) \cdot |f'(z_0)| \leq \frac{1}{1+\epsilon} \rho(z_0)$$

(for any $z_0 \in \Delta$, since $z_0$ was arbitrary).

So integrating this over geodesics connecting points $\alpha, \beta \in \Delta$ yields

$$d_\rho(f(\alpha), f(\beta)) \leq \frac{1}{1+\epsilon} d_\rho(\alpha, \beta)$$

$\Longrightarrow$ $f$ is a contraction mapping of the complete metric space $(\Delta, \rho)$, at which point the result follows from the contraction-mapping fixed point theorem. $\square$