Lecture 21: The Schwarz awakens

I. Proof of IMT/OMT

We will give two proofs of the local mapping theorems, one via the Cauchy integral formula (now) and another via power series methods (at the end).

To begin, let U be a region, $f \in \mathrm{Hol}(U)$ nonconstant with $f(z_0) =: w_0$, and set

$$n := \mathrm{ord}_{z_0}(f(z) - w_0).$$

Let $\epsilon > 0$ be such that (writing $D := D(z_0, \epsilon)$)

$$\begin{cases} f'(z) \neq 0 \\ f(z) \neq w_0 \end{cases} \text{ on } \bar{D}^*,$$

and set $\gamma := \partial D$.

Note: here we are using the fact that zeros of $f'(z)$ and $f(z) - w_0$ can't have z_0 as accumulation point

Theorem $\exists\, \delta > 0$ s.t.

$0 < |\alpha - w_0| < \delta \implies f(z) = \alpha$ has exactly n solutions in D.

Proof: Denote the solutions $\{z_j(\alpha)\}_{j=1}^{\mu(\alpha)}$. Each has multiplicity one, since

$$(f(z)-\alpha)' = f'(z) \neq 0 \quad \text{at } z_j(\alpha). \ (\in D^*)$$

So

$$f(z) - \alpha = \left(\prod_{j=1}^{\mu(\alpha)} (z - z_j(\alpha)) \right) \underbrace{g(z)}_{\left(\text{holo. nonvanishing on } D\right)} ,$$

and

$$\begin{aligned} \mu(\alpha) &= \sum_{j=1}^{\mu(\alpha)} \underbrace{W(\gamma, z_j(\alpha))}_{=1} \\ &= \sum_{j=1}^{\mu(\alpha)} \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_j(\alpha)} + \underbrace{\frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)} dz}_{=0 \text{ by Cauchy}} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f'(z)\, dz}{f(z) - \alpha} \\ &= \frac{1}{2\pi i} \int_{f(\gamma)} \frac{dw}{w - \alpha} , \end{aligned}$$

which is constant in α on components of $\mathbb{C} \setminus f(\gamma)$ (open since $f(\gamma)$ closed). By assumption (that $f(z) \neq w_0$ on ∂D^*) $w_0 \notin f(\gamma)$, and we let V denote the component containing w_0. Take $\delta > 0$ s.t. $D(w_0, \delta) \subset V$, and note $\mu(w_0) = n$; by constancy of the winding number on V, $\mu(\alpha) = n \ \forall \alpha \in D_\delta^*$. $\square$

Corollary 1 (OMT) f sends open sets to open sets

Proof: It suffices to show that for arbitrarily small open disks D about arbitrary $z_0 \in U$, $f(D)$ contains a small open disk. But $f(D) \supset V \supset D(w_0, \delta)$ in the above proof. $\square$

Corollary 2 (IMT) If $f'(z_0) \neq 0$, f is a local analytic $\cong$.

Proof: In this case $n = 1$, and the Theorem implies that f has a single-valued inverse in $D(w_0, \delta)$. This inverse is continuous because f is open. It is then analytic simply by, at each point, taking the reciprocal of the limit computing f' (yielding of course "$\frac{1}{f' \circ f^{-1}}$"). $\square$

Corollary 3 $f : U \to V$ $\begin{cases} \text{analytic,} \\ \text{injective + surjective} \end{cases}$ $\Rightarrow$ f has an analytic inverse.

Proof: f bijective $\Rightarrow$ $\exists$ set-theoretic inverse $g : V \to U$ which is C^0 by OMT. Must show g analytic. Apply Theorem at any point z_0: bijectivity $\Rightarrow$ $n = 1$ $\Rightarrow$ $f'(z_0) \neq 0$. Now apply IMT, done. $\square$

II. Schwarz Lemma

This is a key application of the MMP, which in turn was closely related to the OMT (at least, one of the ways we proved it).

Going one step beyond the "purely local", the next challenge is to identify all the analytic isomorphisms of the unit disk $\Delta = D_1$:

$$\Delta \xrightarrow[\cong]{\Xi} \Delta \quad , \quad \text{i.e. } \text{"}\Xi \in Aut(\Delta)\text{"}.$$

Given a simply connected region U and an analytic isomorphism $f : U \xrightarrow{\cong} \Delta$, you'll then have all automorphisms of U : for each $\xi \in Aut(U)$,

$$\begin{tikzcd} \Delta \arrow[r, "f^{-1}"'] \arrow[rrr, bend left] & U \arrow[r, "\xi"'] & U \arrow[r, "f"'] & \Delta \end{tikzcd}$$

gives an automorphism of Δ, $\Xi = f \circ \xi \circ f^{-1}$. Likewise $\Xi \mapsto f^{-1} \circ \Xi \circ f$ goes the other way, and we get an isomorphism of groups $Aut(U) \cong Aut(\Delta)$. The key result in the computation of $Aut(\Delta)$ is the following (which on the surface only identifies automorphisms <u>fixing the origin</u>):

Schwarz Lemma Given $\Delta \xrightarrow{f} \Delta$ analytic, with $f(0)=0$. Then

(i) $|f(z)| \leq |z|$ $(\forall z \in \Delta)$

(ii) $|f'(0)| \leq 1$

(iii) If $|f(z_0)| = |z_0|$ for any $z_0 \in \Delta^*$, then $f(z) = e^{i\theta} z$ $(\theta \in \mathbb{R})$

(iv) If $|f'(0)| = 1$, then $f(z) = f'(0) \cdot z$.

(i.e. f is a rotation)

Proof: $f \in \mathcal{H}ol(\Delta) \Rightarrow f(z) = \sum_{n \geq 0} a_n z^n$ on (all of) Δ

$$\Rightarrow \limsup |a_n|^{1/n} \leq 1,$$

and $f(0) = 0 \Rightarrow a_0 = 0$. Hence

$$g(z) := \sum_{m \geq 0} a_{m+1} z^m \in \mathcal{H}ol(\Delta)$$

and
$$= \begin{cases} f(z)/z, & z \in \Delta^* \\ f'(0), & z = 0 \end{cases}.$$

i.e. a_1

(i) Show $|g(z)| \leq 1$ $(\forall z \in \Delta)$:

$$f(\Delta) \subseteq \Delta \Rightarrow |f(z)| < 1 \quad (\forall z \in \Delta)$$

⊛ f is not necessarily 1-to-1 or onto.

$$\Rightarrow |g(z)| < \frac{1}{|z|} \quad (\forall z \in \Delta)$$

Let $z_0 \in \Delta$, $r \in [|z_0|, 1)$ be arbitrary. Then $g \in \mathcal{H}ol(\overline{D_r})$, while $\|g(z)\|_{\partial \overline{D_r}} < \frac{1}{r}$. Since $z_0 \in \overline{D_r}$, MMP $\Rightarrow$

$$|g(z_0)| \leq \|g(z)\|_{\partial \overline{D_r}} < \frac{1}{r},$$

and taking $r \to 1^-$, we get $|g(z_0)| \leq 1$.

(ii) $z_0 = 0 \rightsquigarrow |f'(0)| = |g(0)| \leq 1$.

(iii) <u>Show g constant of absolute value 1</u>: $|f(z_0)| = |z_0|$

$\rightarrow |g(z_0)| = 1$. From (i), $\|g\|_\Delta \leq 1$, so max is achieved at $z_0 \underset{\text{MMP}}{\Longrightarrow} g$ constant $= g(z_0)$.

(iv) $|g(0)| = 1 \underset{(iii)}{\Longrightarrow} g \equiv g(0) \Rightarrow f/z \equiv f'(0)$

$$\Rightarrow f(z) = f'(0) z. \qquad \square$$

Now consider the group of conformal automorphisms of Δ, i.e. $\mathrm{Aut}(\Delta)$. Schwarz $\Rightarrow$ <u>any autom. fixing 0 is a rotation</u>: apply (ii) to f & f^{-1}, then apply (iv). What about those not fixing 0? Consider the FLT

$$\phi_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha} z}, \quad \alpha \in \Delta.$$

As a function on Δ this is holomorphic (as the denominator is never 0). Moreover

$$|z|=1 \Rightarrow |\phi_\alpha(z)| = \left|\frac{z-\alpha}{1-\bar{\alpha}z}\right| = \left|\frac{z-\alpha}{\bar{z}(1-\bar{\alpha}z)}\right| \overset{z\bar{z}=1}{=} \left|\frac{z-\alpha}{\bar{z}-\bar{\alpha}}\right| = 1.$$

The MMP says that $|\phi_\alpha| \leq 1$ on Δ, with equality at some point $\Longleftrightarrow$ ϕ_α is constant. Since ϕ_α is visibly <u>**NOT**</u> constant,

$$|z| < 1 \Longrightarrow |\phi_\alpha(z)| < 1.$$

Now writing $w = \frac{z-\alpha}{1-\bar{\alpha}z} \Rightarrow w - \bar{\alpha}zw = z-\alpha \Longrightarrow$
$w+\alpha = z(1+\bar{\alpha}w) \Rightarrow z = \frac{w+\alpha}{1+\bar{\alpha}w} = \phi_{-\alpha}(w)$
we find that

$$\boxed{\phi_\alpha^{-1} = \phi_{-\alpha}.}$$

Conclude that

$$\phi_\alpha \in \mathrm{Aut}(\Delta).$$

Another type of element of $\mathrm{Aut}(\Delta)$ is $\mu_{e^{i\theta}}$, where

$\mu_{e^{i\theta}}(z) := e^{i\theta} z \quad (\theta \in \mathbb{R}).$

$\boxed{\text{Theorem A}}$ $F \in \mathrm{Aut}(\Delta) \Rightarrow \exists\, \alpha \in \Delta,\ \theta \in \mathbb{R}$ s.t. $F = \phi_\alpha \circ \mu_{e^{i\theta}} \left(= \mu_{e^{i\theta}} \circ \phi_{\alpha e^{-i\theta}}\right)$.

Proof: Use the Schwarz:

let $\beta := F(0)$; applying Schwarz (ii) to $f := \phi_\beta \circ F$ (which sends $0 \mapsto 0$), $|f'(0)| \leq 1$. Applying it to $f^{-1} = F^{-1} \circ \phi_{-\beta}$, $\left|\frac{1}{f'(0)}\right| = |(f^{-1})'(0)| \leq 1$. So $|f'(0)| = 1 \underset{\text{Schwarz (iv)}}{\Longrightarrow} (\phi_\beta \circ F =)\, f = \mu_{e^{i\theta}}$ for some θ

$\Rightarrow F = \phi_{-\beta} \circ \mu_{e^{i\theta}}$. $\square$

So we've proved that

$$\mathrm{Aut}(\Delta) \cong \left\{ e^{i\theta} \frac{z-\beta}{1-\bar{\beta}z} \;\middle|\; \beta \in \Delta,\ \theta \in \mathbb{R} \right\}.$$

$\boxed{\text{Theorem B}}$ $\mathrm{Aut}(\mathfrak{h}) \cong \{ f_M \mid M \in SL_2(\mathbb{R}) \}$ $(\cong PSL_2(\mathbb{R}))$

<u>Remark</u>: In HW, you showed that <u>amongst FLTs</u> the ones giving automorphisms of $\mathfrak{h}$ were of this form, and that (conversely) all f_M's of this form <u>are</u> Auts of $\mathfrak{h}$.

Proof: Let $g \in \mathrm{Aut}(\mathfrak{h})$, so $x+iy := g(i) \in \mathfrak{h}$
$\Rightarrow y > 0$. Write

$$M = \begin{pmatrix} 1/\sqrt{y} & -x/\sqrt{y} \\ 0 & \sqrt{y} \end{pmatrix} \in SL_2(\mathbb{R}) \quad ;$$

then

$$f_M(g(i)) = \frac{(1/\sqrt{y})(x+iy) - x/\sqrt{y}}{\sqrt{y}} = \frac{x+iy}{y} - \frac{x}{y} = \frac{iy}{y} = i.$$

Set $h := f_M \circ g : \begin{array}{c} \mathfrak{h} \xrightarrow[\cong]{} \mathfrak{h} \\ i \longmapsto i \end{array}$. (It's an $\cong$ because both f_M & g are.)

Let $F(z) := \frac{z-i}{z+i} = f_N$, $N = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$, $N^{-1} = \frac{1}{2i}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$.

Then $f_N = F : i \mapsto 0$ and $f_{N^{-1}} = F^{-1} : 0 \mapsto i$, so

$$F \circ h \circ F^{-1} : \Delta \xrightarrow[\cong]{} \Delta$$

takes $\qquad 0 \longmapsto 0$

hence must equal $\mu_{e^{i\theta}}$ (by the remark right after proof of Schwarz). So

$$h = F^{-1} \circ \mu_{e^{i\theta}} \circ F = f_{N^{-1}} \circ \underbrace{f_{\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}}}_{=: Q} \circ f_N$$

$$\frac{e^{i\theta}z+0}{0z+1} = \frac{e^{i\theta/2}z+0}{0z+e^{-i\theta/2}}$$

$$= f_{\underbrace{N^{-1}QN}_{=:M'}} = f_{\begin{pmatrix} \cos(-\theta/2) & -\sin(-\theta/2) \\ \sin(-\theta/2) & \cos(-\theta/2) \end{pmatrix}}$$

$\Rightarrow M' \in SL_2(\mathbb{R})$

$\Rightarrow M^{-1}M' \in SL_2(\mathbb{R})$

and

$$g = f_{M^{-1}} \circ h = f_{M^{-1}M'} .$$

$\square$

Next comes a refinement of Schwarz beyond the $0 \mapsto 0$ case:

$\boxed{\text{Schwarz - Pick Theorem}}$ Let $F : \Delta \to \Delta$ be holomorphic, and $z_1 \& z_2 \in \Delta$ distinct. Then

$$\left| \frac{F(z_1) - F(z_2)}{1 - F(z_1)\overline{F(z_2)}} \right| \leq \left| \frac{z_1 - z_2}{1 - z_1 \overline{z_2}} \right| , \text{ and } |F'(z_1)| \leq \frac{1 - |F(z_1)|^2}{1 - |z_1|^2} .$$

If for any z_1, z_2 one gets equality in either expression, then $F \in \mathrm{Aut}(\Delta)$.

Proof: $\phi_{F(z_1)} \circ F \circ \phi_{-z_1} \in \mathcal{H}ol(\Delta)$ and sends

$$0 \xrightarrow[\phi_{-z_1}]{} z_1 \xrightarrow[F]{} F(z_1) \xrightarrow[\phi_{F(z_1)}]{} 0 .$$

So Schwarz (i) $\Rightarrow$ $\left|\left(\phi_{F(z_1)} \circ F \circ \phi_{-z_1}\right)(z)\right| \leq |z|$.

$$\Longrightarrow \quad \left|\phi_{F(z_1)}(F(z_2))\right| \leq \left|\phi_{z_1}(z_2)\right| .$$

writing $z = \phi_{z_1}(z_2)$ $\qquad$ $\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}$

Also, Schwarz (ii) $\Rightarrow$

$$1 \geq \left|\left(\phi_{F(z_1)} \circ F \circ \phi_{-z_1}\right)'(0)\right| = \left|\phi'_{F(z_1)}(F(z_1))\right| \left|F'(z_1)\right| \left|\phi'_{-z_1}(0)\right|$$

$$= \frac{\cancel{1 - |F(z_1)|^2}}{(1 - |F(z_1)|^2)^{\cancel{2}}} \cdot |F'(z_1)| \cdot \frac{1 - |z_1|^2}{1}$$

(where we have used $\phi'_\alpha(z) = \frac{(1 - \bar{\alpha} z) + \bar{\alpha}(z - \alpha)}{(1 - \bar{\alpha} z)^2} = \frac{1 - |\alpha|^2}{(1 - \bar{\alpha} z)^2}$).

Finally, equality (in the statement of the Thm.)

$\Rightarrow$ equality in one of the circled inequalities

$$\underset{\text{Schwarz (iii)-(iv)}}{\Longrightarrow} \quad \phi_{F(z_1)} \circ F \circ \phi_{-z_1} = \mu_{e^{i\theta}}$$

$$\Longrightarrow \quad F = \phi_{-F(z_1)} \circ \mu_{e^{i\theta}} \circ \phi_{z_1} \quad (\in \mathrm{Aut}(\Delta)) .$$

$\Box$

III. Proof of IMT/OMT (bis)

Here is the approach by power series, which uses the calculation in (II) from Lecture 20. Using the notation there, if f has nonzero radius of conv. r, then $|a_n| \leq A^n$ $(\forall n)$. We want $r_g > 0$.

Set $F(T) := T - \sum_{n \geq 2} A^n T^n$, with formal composition inverse $G(T) := \sum B_k T^k$. We have

$$B_k = \sum_{\substack{I, \vec{r} \\ \left(\sum i \vec{r}(i) = k, \ \sum \vec{r}(i) \geq 2\right)}} A^{\sum_{i \in I} \vec{r}(i)} \left(\prod_{i \in I} B_i^{\vec{r}(i)} \right) \frac{(\sum \vec{r}(i))!}{\vec{r}(i_1)! \cdots \vec{r}(i_{|I|})!}$$

$$\geq \sum_{I, \vec{r}} \left| a_{\sum_I \vec{r}(i)} \right| \left(\prod_I |b_i|^{\vec{r}(i)} \right) \frac{(\sum \vec{r}(i))!}{\vec{r}(i_1)! \cdots \vec{r}(i_{|I|})!}$$

(inductively assuming $B_\ell \geq |b_\ell|$ $\forall \ell < k$.)

$$\geq \Bigg| \underbrace{\sum_{I, \vec{r}} a_{\sum_I \vec{r}(i)} \left(\prod_I b_i^{\vec{r}(i)} \right) \frac{(\sum \vec{r}(i))!}{\vec{r}(i_1)! \cdots \vec{r}(i_{|I|})!}}_{b_k} \Bigg|$$

(see Lecture 20)

So $\limsup |B_k|^{1/k} \geq \limsup |b_k|^{1/k}$

$\Rightarrow r_g \geq r_G \Rightarrow$ suff. to show $r_G > 0$.

Now (working formally)

$$T = (F \circ G)(T) = G - \sum_{n \geq 2} (AG)^n$$

$$= G - \frac{A^2 G^2}{1 - AG}$$

by defn.

$$\Rightarrow \quad 0 = (A^2 + A) G^2 - (1 - AT) G + T$$

mult. by $(1-AG)$

$$\Rightarrow \quad G(T) = \frac{(1+AT)}{2A(1+A)} \left\{ 1 - \left(1 - 4T \frac{(A^2+A)}{(1+AT)^2} \right)^{1/2} \right\}.$$

"solve" quadratic equation (understood formally, in sense of series solving equation)

composition of formal power series with POSITIVE radii of convergence

$$r_G > 0.$$

Now we apply this to functions $f(z), g(z)$. (Recall these are analytic / continuous.) Main idea: $(f \circ g)(T) = T$ $\Rightarrow f(g(z)) = z$, etc. By continuity, $\exists\, D_\epsilon(0) = V$ s.t. $g(V) \subset D_{r_f}(0)$. Put $U := f^{-1}(V)$, so $f : U \to V$. Given $v \in V$, $f(g(v)) = v \Rightarrow g(v) \in f^{-1}(V) = U$, so $g : V \to U$ and have f, g local analytic isomorphisms at 0. This proves "$\Leftarrow$" of the IMT. cf. Lect. 19(III)

Now let $\mathrm{ord}(f) = m \geq 1$. (Until now, it's been $=1$.)

That is, $f(z) = a_m z^m (1 + h(z))$, $\quad h(0) = 0$, $\quad 0 \neq a_m = \alpha^m$

$$= [\alpha z (\underbrace{1 + h_1(z)}_{\text{from expansion of } (1+h)^{1/m}})]^m$$

$$= (f_1(z))^m,$$

where $f_1(z) = \underset{\substack{\neq \\ 0}}{\alpha} z + \{\text{higher-order terms}\} \underset{\text{from above}}{\Rightarrow} f_1$ local analytic iso. at 0

$\Rightarrow f_1$ open near 0 $\Rightarrow f$ open (since $w \mapsto w^m$ is too).

(use continuity of the inverse)

Any nonconstant f (taking $0 \mapsto 0$) is of the form above, so this proves $\underbrace{\text{"}\Leftarrow\text{" of OMT}}_{\text{cf. Lect. 19 (III)}}$ ("$\Rightarrow$" is trivial).

If $m > 1$, then f cannot (locally) be an injection, since $w \mapsto w^m$ isn't; hence f will not be locally invertible, proving "$\Rightarrow$" of IMT.

Appendix

We conclude with a sample use of the formula for the coefficients $\{b_k\}$ of the inverse power series of

$$f(z) = \sum_{n \geq 1} a_n z^n, \quad a_1 = 1:$$

- $b_1 = 1$, and

$$\bullet\ -b_k = \sum_{\substack{I,\vec{\mu} \\ \left(\begin{smallmatrix}\sum_{i\in I}\vec{\mu}(i)\geq 2 \\ \sum_{i\in I} i\vec{\mu}(i) = k\end{smallmatrix}\right)}} a_{\sum_I \vec{\mu}(i)} \left(\prod_I b_i^{\vec{\mu}(i)}\right) \frac{(\sum_I \vec{\mu}(i))!}{\vec{\mu}(i_1)! \cdots \vec{\mu}(i_{|I|})!}$$

Think of $\sum_I i\vec{\mu}(i)$ as a partition of k. ($\vec{\mu}$ is "frequency")

This is ugly, but sometimes more efficient than the alternative (writing out $z = (f\circ g)(z)$).

Example

$f(z) = \sin(z)$.

$a_2 = 0,\ a_3 = -\frac{1}{6},\ a_4 = 0,\ a_5 = \frac{1}{120}, \dots$

Obviously $b_{\text{even}} = 0$.

$$b_3 = -a_3 b_1^3 \frac{3!}{3!} = -\left(-\frac{1}{6}\right) 1^3 \cdot 1 = \frac{1}{6} = \frac{1}{2\cdot 3}.$$

only partition into odd #'s (smaller) is $1+1+1=3$.

$$b_5 = -a_5 b_1^5 \frac{5!}{5!} - a_3 b_1^2 b_3 \frac{3!}{2!\,1!} = \frac{-1}{120} + \frac{1}{6}\cdot\frac{3}{6} = \frac{3}{40} = \frac{1\cdot 3}{2\cdot 4\cdot 5}$$

$5 = 1+1+1+1+1 = 3+1+1$

etc.