Lecture 20 : Applications of Cauchy

I. Consequences of MMP

In Lecture 19 we used Cauchy Theorem/integral formula to prove the Maximum Modulus Principle (that a holomorphic function on a region in $\mathbb{C}$ which attains a maximum $|\cdot|$ there, must be constant).

connected open set

Here are some immediate consequences.

Corollary 1 Given $K \subseteq \mathbb{C}$ compact, $f : K \to \mathbb{C}$ continuous & nonconstant, with $f|_{\mathring{K}}$ analytic and $\mathring{K}$ connected. Then $|f|$ attains its maximum over K on ∂K.

Proof: $K \xrightarrow{f} \mathbb{C} \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0}$ is a continuous function on a compact set, hence attains $\sup_K |f|$

at some point $z_0 \in K$. Suppose $z_0 \in \mathring{K}$.

Then by MMP $f|_{\mathring{K}}$ is constant $\underset{f\ C^0}{\Longrightarrow}$ f constant. $\square$

We also get a 2^{nd} proof of the Fundamental Theorem of Algebra:

see also Lecture 18

$\boxed{\text{Corollary 2}}$ Any nonconstant polynomial ($\underset{\#}{0}$, $n>0$)

$P(z) = a_n z^n + \cdots + a_0$

has a root $\in \mathbb{C}$.

Proof: Assume otherwise; then $\dfrac{1}{P}$ is entire.

Now $\displaystyle\lim_{|z|\to\infty} \frac{1}{P(z)} = 0 \Rightarrow \exists\, R>0$ s.t.

$|z| \geq R \Rightarrow \left|\dfrac{1}{P(z)}\right| < \left|\dfrac{1}{P(0)}\right|$.

By Cor. 1, $\left|\dfrac{1}{P}\right|$ must either

- be constant on $\overline{D_R}$

or

- attain its maximum on $\overline{D_R}$ in ∂D_R.

Clearly, both are impossible ($\Rightarrow \exists$ a root). $\square$

<u>Remark</u>: I didn't explain before why $P(z)$ then equals $C \cdot \prod_{i=1}^{n} (z - z_i)$. P has root at $z_0 \Rightarrow$ $P(z_0) = 0$. By division algorithm,

$$P(z) = (z - z_0)\, g_0(z) + P_1(z), \quad \deg P_1 < \deg(z - z_0) = 1$$

$$\big(\overset{\shortparallel}{0}\big)$$

$\Rightarrow P_1(z) = C_1$, and substituting in z_0 gives $0 = P(z_0) = P_1(z_0) = C_1$.

Now keep going, with g_0 replacing P. //

Corollary 3 Let $\{z_i\}_{i=1}^{n}$ be points on the boundary of the unit disk Δ. Then the product of the distances $d(z, z_i)$ is at least 1 somewhere on the unit circle.

Proof: Set $f(z) = \prod_{i=1}^{n} (z - z_i)$. Then $|f(0)| = 1$, and f is obviously not constant. By Cor. 1, $\max_{z \in \overline{\Delta}} |f(z)|$ is attained on $\partial \Delta$. $\square$

(The HW will contain more applications.)

II. Composition inverse

READING ASSIGNMENT!

Given $f(T) = T + 12T^2 - 60T^3 + \cdots$ (say),

we want $g(T) = T + b_2 T^2 + b_3 T^3 + \cdots$ s.t.

$$T = (f \circ g)(T) := g(T) + 12(g(T))^2 - 60(g(T))^3 + \cdots$$
$$= T + \underbrace{(b_2 + 12)}_{=0} T^2 + \underbrace{(b_3 + 24b_2 - 60)}_{=0} T^3 + \cdots$$

Clearly we must take $b_2 = -12$, $b_3 = 348$.

More abstractly, if $f(T) = T - \sum_{n \geq 2} a_n T^n$,

(want corresponding fcn. to have $f'(0) \neq 0$.)

write
$$T = (f \circ g)(T) = \sum_{k \geq 1} b_k T^k - \sum_{n \geq 2} a_n \Big(\sum_{k \geq 1} b_k T^k \Big)^n,$$

where $\sum_{n \geq 2} a_n \Big(\sum_{k \geq 1} b_k T^k \Big)^n$

$$= \sum_{n \geq 2} a_n \left\{ \sum_{I, \vec{\mu}} \Big(\prod_{i \in I} b_i^{\vec{\mu}(i)} \Big) \frac{n!}{\vec{\mu}(i_1)! \cdots \vec{\mu}(i_{|I|})!} T^{\sum_{i \in I} i \vec{\mu}(i)} \right\}$$

where $\vec{\mu} : I \to \mathbb{N}$
$I = \{i_1, \ldots, i_{|I|}\}$
and $\sum_{i \in I} \vec{\mu}(i) = n$.

$$= \sum_{n\geq 2} a_n \sum_{k\geq 2} \sum_{\substack{I,\tilde{j} \\ \left(\sum_I \tilde{j}(i) = n \atop \sum_I i\,\tilde{j}(i) = k\right)}} \left(\prod_I b_i^{\tilde{j}(i)} \right) \frac{n!}{\tilde{j}(i_1)! \cdots \tilde{j}(i_{|I|})!} T^k$$

finite sum

$$= \sum_{k\geq 2} T^k \left\{ \sum_{\substack{I,\tilde{j} \\ \left(\sum_I i\,\tilde{j}(i) = k \atop \sum_I \tilde{j}(i) \geq 2\right)}} a_{\sum_I \tilde{j}(i)} \left(\prod_I b_i^{\tilde{j}(i)} \right) \frac{\left(\sum \tilde{j}(i)\right)!}{\tilde{j}(i_1)! \cdots \tilde{j}(i_{|I|})!} \right\}$$

ensures that every $i \in I$ is <u>less</u> than k $\begin{pmatrix} * \\ * \end{pmatrix}$

Defining b_k inductively to be the quantity in red braces (possible by $\begin{pmatrix} * \\ * \end{pmatrix}$) then makes $(f \circ g)(T) = T$.

We have then (formally)

$$g(f(T)) = g(\underbrace{f(g}_{id}(h(T)))) = g(h(T)) = T$$

write $h(T)$ for g's <u>right</u> composition inverse

So g gives a 2-sided formal composition inverse to f (and $f = h$). In the next Lecture, we will use this to prove IMT/OMT.

III. Cycles and connectivities

Let U be an open set in $\mathbb{C}$ which is the union of finitely many regions.

Recall that, denoting C^1 i-chains by $C_i(U)$, we have boundary homomorphisms (with $\partial \circ \partial = 0$)

$$0 \xrightarrow{\partial} C_2(U) \xrightarrow{\partial} C_1(U) \xrightarrow{\partial} C_0(U) \xrightarrow{\partial} 0$$

and $\underbrace{H_i(U,\mathbb{Z})}_{\text{or just "}H_i(U)\text{"}} := \dfrac{\ker(\partial)}{\operatorname{im}(\partial)}$ at the i^{th} place.

(singular homology groups)

An i-chain in $\ker(\partial)$ is closed, or an i-cycle.

Let $h_i := \operatorname{rank}(H_i)$.

Proposition 1 Let $c := \#\{\text{connected components of } U\}$. Then $h_0 = c$.

Proof: Define a map

$$H_0(U) \xrightarrow{\beta} \mathbb{Z}^c \qquad (U = U_1 \amalg \cdots \amalg U_c)$$

by $[p] \xmapsto{p \in U_i}$ basis vector $e_i = (0,\dots,\underset{i}{1},\dots,0)$

Check that this map is

- well-defined: $\underbrace{\gamma \text{ path with } \partial\gamma = [p]-[q]}_{(\text{generators of } \underset{\text{hom}}{\equiv})} \Rightarrow p, q \in \text{same } U_i$

- injective: $\beta\left(\sum n_p[p]\right) = 0 \Rightarrow \sum_{p \in U_i} n_p = 0 \ (\forall i)$

$$\Rightarrow \exists\, \gamma \text{ with } \partial\gamma = \sum_{p \in U_i} n_p[p]$$

("connect the dots")

- surjective: obvious

$\Rightarrow \beta$ is an isomorphism, done. $\square$

Conclude that

$$U \text{ is connected} \iff h_0 = 1.$$

Assume now that U is connected, and consider h_1.

$\boxed{\text{Proposition 2}}$ Let $n := \#\{\text{connected components of } U^c\}$ (=: "connectivity" of U). Then $h_1 = n - 1$.

Proof: Write $U^c = \underbrace{V_1 \sqcup \cdots \sqcup V_{n-1}}_{\text{bounded}} \sqcup \underset{\text{unbounded}^\dagger}{V_n}$

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† If we took the complement of U in $\hat{\mathbb{C}}$, then $V_n \cup \{\infty\}$ is the component containing ∞.

and take (for each i) $a_i \in V_i$ any point. Define

$$H_1(U) \xrightarrow{\beta} \mathbb{Z}^{n-1}$$

by

$$[\gamma] \longmapsto (W(\gamma, a_1), \dots, W(\gamma, a_{n-1})).$$

Check that β is

- well-defined: $\gamma = \partial K \implies W(\gamma, \alpha) = 0 \ \forall \alpha \in U^c$.

  (cf. (*) from lecture 18)

- surjective: cover $\mathbb{C}$ with squares sufficiently small that no square meets more than one V_i. (Let $\mu = \min_{i \neq j} d(V_i, V_j)$, take side-length $\frac{1}{4}\mu$.)

  $\mu > 0$ as the V_i are compact

  Let $\mathcal{W}_i$ = union of (closed) squares meeting V_i, then $V_i \subset \mathcal{W}_i$ while $\partial \mathcal{W}_i \cap V_i = \emptyset = \mathcal{W}_i \cap V_{j \neq i}$.

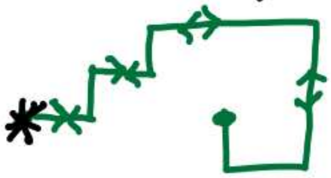
  Consequently $\partial \mathcal{W}_i \subset U$ and, by considering integrals over the squares in $\mathcal{W}_i$,

$$W(\partial \mathcal{W}_i, a_j) = \delta_{ij}.$$

  So the image of β hits all the e_i vectors and we're done.

- injective: Say $W(\gamma, a_i) = 0$ ($\forall i$). Then $W(\gamma, \alpha) = 0$ for any α in any V_i, since W is constant on connected components of $\mathbb{C} \setminus \gamma$ (and each V_i is contained in one such component). Now use

$W(\gamma, \alpha) = 0 \ \forall \alpha \in U^c \Rightarrow \gamma = \partial K$ (again using (*) from lecture 18). $\square$

<u>Remark</u>: Suppose $n = 1$ and $\gamma \in \pi_1(U, *)$ is a loop. Subdivide the plane into rectangles as above, and WOLOG assume γ is rectangular. Then the union of the bounded components of the complement of $\gamma([a,b])$ in $\mathbb{C}$ is a "union of squares" $\mathcal{S} = \cup R_i \subset U$. Since $\partial R_i \sim 0$, we can "remove" squares from the outside of $\mathcal{S}$ until $\mathcal{S}$ is empty, moving γ (by homotopies) so that $\mathcal{S}$ remains the union of bounded components of its complement. But $\mathcal{S} = \emptyset$ $\Rightarrow \gamma^c =$ connected open $\subset \mathbb{C}$, and the image of γ is something like , which is easily homotopied to $*$. //

Conclude from the Remark + Hurewicz $(\pi_1 \twoheadrightarrow H_1)$ that

$$U \text{ simply connected} \Longleftrightarrow \pi_1 = \{0\} \Leftrightarrow h_1 = 0.$$

In the proof of the 2nd Prop., we saw that

$H_1(U)$ = $\underline{\text{free } \underline{\text{abelian}} \text{ group}}$ on $n-1$ generators $[\gamma_i]$

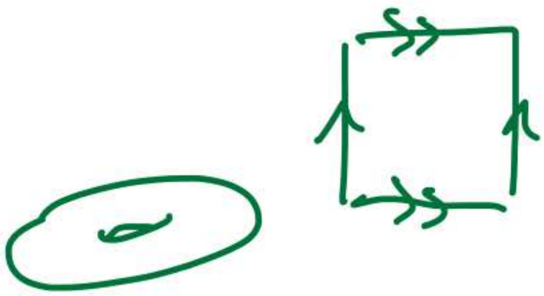
(We also have, but won't prove, $\pi_1(U)$ = free group on the γ_i.)† Hence, for any closed path γ in U

(*) $$\underbrace{\gamma \underset{\text{hom}}{\equiv} \sum_{i=1}^{n-1} a_i \gamma_i}_{(\text{in } U)}, \quad \text{for some } a_i \in \mathbb{Z}.$$

Finally, $H_2(U) = \{0\}$, because there are no closed finite sums of triangles. Otherwise, the union of these triangles would cover U, hence U would be compact, a contradiction. To get $H_2(U)$ nonzero, you first need to replace U by a compact real 2-manifold M (e.g. a Riemann surface). Then M is orientable $\underset{\text{def.}}{\Longleftrightarrow}$ $H_2(M) \neq \{0\}$ and is spanned by the fundamental class $[M]$ (i.e. the class of a finite triangulation of M).

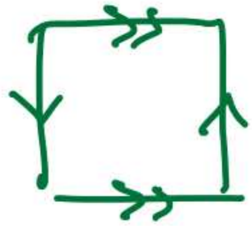
† i.e. elements of π_1 are the words in $n-1$ generators. For compact Riemann surfaces, $H_1(M) = \mathbb{Z}\langle \gamma_1, \ldots, \gamma_{2g} \rangle$, and $\pi_1(M) \cong \{\text{free group on } \gamma_1, \ldots, \gamma_{2g}\} / \text{relation } [\gamma_1, \gamma_{g+1}] \cdot [\gamma_2, \gamma_{g+2}] \cdots [\gamma_g, \gamma_{2g}] \sim 1$.

Example // Consider the manifolds obtained by identifying sides of the square as shown:



Torus

$h_2 = 1$



Klein bottle

$h_2 = 0$. //

IV. Periods of closed differentials

So we have now had a more thorough look at chains and homology. Next we have to examine differential forms and (de Rham) cohomology. Denoting

$$A^k(U) := C^\infty \text{ } k\text{-forms on } U$$

the exterior derivative (cf. Lecture 14) gives homomorphisms

$$0 \xrightarrow{d} \underbrace{A^0(U)}_{C^\infty \text{ functions}} \xrightarrow{d} \underbrace{A^1(U)}_{\substack{\text{expressions} \\ f\,dx + g\,dy \\ (\text{or } F\,dz + G\,d\bar{z})}} \xrightarrow{d} \underbrace{A^2(U)}_{\substack{\text{expressions} \\ H\,dx \wedge dy \\ \text{or } dz \wedge d\bar{z}}} \xrightarrow{d} 0$$

(f, g, F, G, H all in $C^\infty(U)$)

Note in particular that $d \circ d = 0$:

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy \quad , \quad d(f\,dx + g\,dy) = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$

$$\Rightarrow d(dF) = \left(\frac{\partial^2 F}{\partial x \partial y} - \frac{\partial^2 F}{\partial y \partial x}\right) dx \wedge dy = 0 .$$

Define the de Rham cohomology groups of U by

$$H^k_{dR}(U) := \frac{\ker(d)}{\operatorname{im}(d)} \quad \text{at the } k^{\text{th}} \text{ spot.}$$

Clearly

$$H^0_{dR}(U) = \{\text{constant functions}\} \cong \mathbb{C}$$

$$H^2_{dR}(U) = \{0\} \quad \text{because all the 2-forms are exact}$$
(won't prove).

Now $\ker(d) \subset A^1(U)$ are (by definition) the closed 1-forms:

$$\omega = A(x,y)\,dx + B(x,y)\,dy = f(z)\,dz + g(z)\,d\bar{z}$$

C^∞ fcns.

such that

$$0 = d\omega = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy = \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right) dz \wedge d\bar{z}$$

(i.e. $\partial B/\partial x = \partial A/\partial y$ or [equivalently] $\partial g/\partial z = \partial f/\partial \bar{z}$). In particular, $f(z)\,dz$ is closed $\iff f \in \mathrm{Hol}(U)$ (holomorphic 1-forms).

Poincaré Lemma Closed 1-forms are the same as locally exact 1-forms, i.e. those with a (not necessarily holomorphic) primitive in a small disk about each point.

(not same as global exactness, i.e. being in $d(A^0(U))$)

Proof: (1) ω locally exact

$$\Rightarrow \omega \underset{\text{loc}}{=} \underset{C^\infty}{dF} = \underbrace{\frac{\partial F}{\partial z}}_{f} dz + \underbrace{\frac{\partial F}{\partial \bar z}}_{g} d\bar z$$

$$\Rightarrow \frac{\partial f}{\partial \bar z} = \frac{\partial g}{\partial z}$$

$\Rightarrow$ ω closed.

(2) ω closed

$\Rightarrow$ $F := \int_{z_0} \omega$ is well-defined on a small disk, because (for $\gamma = \partial \Gamma$, $\Gamma \subset U$)

$$\int_\gamma A\,dx + B\,dy = \int_\Gamma \underbrace{\cancel{\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)}}_{0 \text{ by closedness}} dx\,dy = 0.$$

You can check by FTC that $dF = A\,dx + B\,dy = \omega$.

$\square$

Theorem The pairing $H_1(U) \times H^1_{dR}(U) \to \mathbb{C}$

$$([\gamma], [\omega]) \longmapsto \int_\gamma \omega$$

is well-defined.

[That is, $\int_\gamma \omega = 0$ if (i) $\gamma = \partial K$ (i.e. $\gamma \underset{\text{hom}}{\equiv} 0$) or (ii) $\omega = dG$ (ω globally exact).]

Proof†: (i) Triangulate K, so that $\gamma = \sum m_i \partial \tau_i$; use local exactness of ω to see $\int_{\partial \tau_i} \omega = 0$ (as in proof of Poincaré).

(ii) Multivariate fundamental theorem of calculus:

$$\int_\gamma dG = G(\gamma(b)) - G(\gamma(a)) = 0 \quad \text{since } \gamma(b) = \gamma(a)$$

(as γ is closed). $\square$

Letting $a_i \in V_i$ and γ_i be as in §III, consider the forms $\omega_j = \frac{1}{2\pi i} \frac{dz}{z - a_j}$. Since

$$\int_{\gamma_i} \omega_j = W(\gamma_i, a_j) = \delta_{ij},$$

we see that the pairing

$$\underbrace{H_1(U, \mathbb{C})}_{H_1(U) \otimes_{\mathbb{Z}} \mathbb{C}} \times H^1_{dR}(U) \to \mathbb{C}$$

is in fact perfect, so that

$$H^1(U, \mathbb{C}) := \mathrm{Hom}(H_1(U, \mathbb{Z}), \mathbb{C}) \cong H^1_{dR}(U).$$

This is a special case of de Rham's Theorem.

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† or, if you prefer, using Stokes:

$$\int_{\partial K} \omega = \int_K \overset{0}{\cancel{d\omega}} = 0 \; ; \quad \int_\gamma dG = \int_{\underset{0}{\cancel{\partial\gamma}}} G = 0.$$

which says more generally for a smooth manifold M that

$$\underbrace{H^k(M,\mathbb{C})}_{\substack{\text{i.e. } \mathrm{Hom}(H_k(M),\mathbb{C}) \\ \text{(singular cohomology)}}} \cong \underbrace{H^k_{dR}(M)}_{\text{de Rham cohomology}} .$$

The upshot of all this (for our purposes) is the extremely simple consequence of (i) in the above Theorem (applied to $\gamma - \sum n_i \gamma_i$) :

$\boxed{\text{Corollary}}$ Let U be a region with connectivity n, so that $H_1(U) \cong \mathbb{Z}\langle [\gamma_1], \ldots, [\gamma_{n-1}] \rangle$ (free abelian group on $n-1$ generators). Then for any 1-cycle $\gamma \subset U$ and closed 1-form $\omega \in A^1(U)$, we have

$$\gamma \underset{\text{hom}}{\equiv} \sum_{i=1}^{n-1} a_i \gamma_i \quad \text{for some integers } \{a_i\}_{i=1}^{n-1}$$

and

$$\int_\gamma \omega = \sum_{i=1}^{n-1} a_i \int_{\gamma_i} \omega .$$

This says that to know any closed-path integral of ω, you need only know its periods $\int_{\gamma_i} \omega$

over a homology basis. Riemann initiated the systematic study of periods on algebraic curves, and the modern theory most closely associated with it is Hodge theory. There is also a fascinating article (available online) called "Periods" by M. Kontsevich and D. Zagier, which shows how they are really at the center of modern number theory as well.