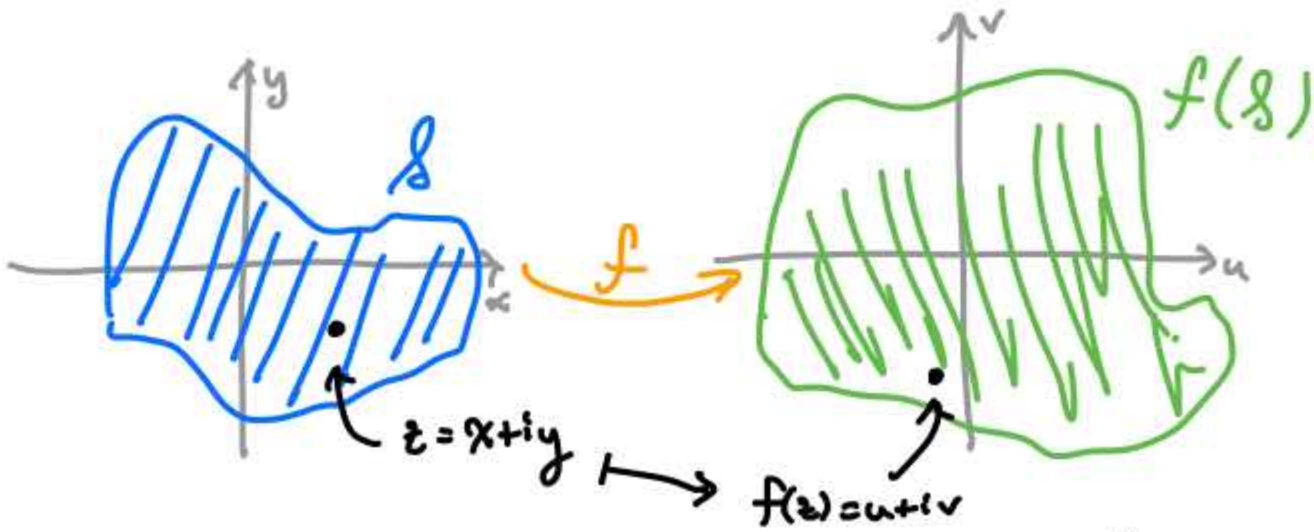
Lecture 2 : Complex functions

I. Some examples

We'll consider complex-valued functions

$$f : S \to \mathbb{C}$$

defined on a subset $S \subseteq \mathbb{C}$.



The interplay between "$\mathbb{C} \to \mathbb{C}$" and "$\mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$" is especially important here. To be able to discuss this clearly we shall fix the notation

$$\underbrace{f(x+iy)} = f(z) = u(z) + iv(z) = \underbrace{u(x,y) + iv(x,y)}$$

for this lecture.

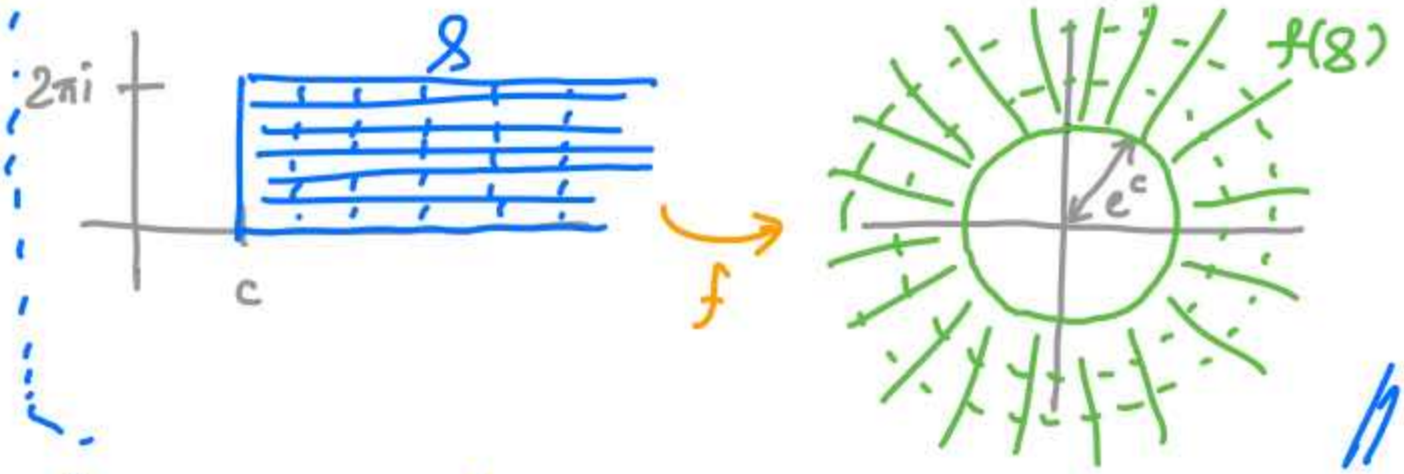
Example 1 (Exponential map)

Define $f(z) = e^z := e^x e^{iy} = \underbrace{e^x \cos y}_{u(x,y)} + i\underbrace{e^x \sin y}_{v(x,y)}$

If $S = \{x+iy \mid x \geq c,\ 0 \leq y \leq 2\pi\} \subseteq \mathbb{C}$,

then $f(S) = \{z \mid |z| \geq e^c\}$, as $e^x \geq e^c$ and

e^{iy} takes all angles when $y \in [0, 2\pi]$.



Example 2

$f(z) = \dfrac{1}{(\bar{z})} = \dfrac{1}{x-iy} = \dfrac{x+iy}{x^2+y^2}$

"preserves angle but inverts length"

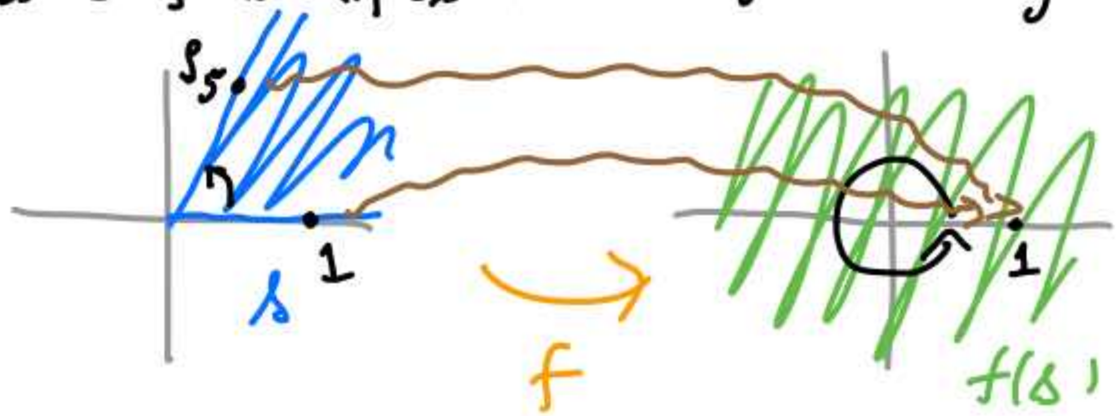
$= \underbrace{\dfrac{x}{x^2+y^2}}_{u(x,y)} + i\underbrace{\dfrac{y}{x^2+y^2}}_{v(x,y)} = \dfrac{1}{r}\underbrace{\left(\dfrac{x}{r} + i\dfrac{y}{r}\right)}_{\cos + i\sin} = \dfrac{1}{r}e^{i\theta}$

(flips about unit circle)

Example 3 $f(z) = z^5 = r^5 e^{i(5\theta)}$.

$u(z) = r^5 \cos(5\theta)$, $v(z) = r^5 \sin(5\theta)$.

(less easy to express in terms of x and y ...)



II. Complex differentiability

For a real function of a real variable

$$g : \underset{\substack{\cap \text{ open} \\ \mathbb{R}}}{V} \longrightarrow \mathbb{R},$$

recall the properties of being

(a) differentiable

(b) infinitely differentiable ("smooth" or C^∞)

(c) analytic (locally representable by power series)

are all distinct.

Example 4

$$g(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x^2} & x > 0 \end{cases}$$ is smooth but non-analytic

(all derivatives are zero at $x = 0$).

For a complex function

$$f : U \to \mathbb{C},$$

$U \subset \mathbb{C}$ open

the properties (to be defined) of being

(a) complex differentiable ("holomorphic")

(b) infinitely complex differentiable (not called smooth)

(always refers to infinite real differentiability)

(c) [complex] analytic (power series in complex variable)

will turn out to, in contrast, be equivalent.

Let's start from a multivariable (real) calculus perspective: suppose we have given

$$\vec{F} : \underset{\mathbb{R}^2 \supset \text{ open}}{U} \longrightarrow \mathbb{R}^2,$$

and write $\vec{F}\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{z}} = \begin{pmatrix} u \\ v \end{pmatrix}$. This is differentiable

(in the real 2-variable $\leftrightarrow$ 2-variable sense) at $\vec{z}_0 \in U$,

iff $\exists\, A \in M_2(\mathbb{R})$ such that

matrix multiplication

$$(*) \quad \boxed{\lim_{\vec{h} \to \vec{0}} \frac{\| \vec{F}(\vec{z}_0 + \vec{h}) - \vec{F}(\vec{z}_0) - A \cdot \vec{h} \|}{\| \vec{h} \|} = 0}$$

Now $(*) \Rightarrow$ existence of the 4 partials u_x, u_y, v_x, v_y at z_0,

and in fact

$$A = \begin{pmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{pmatrix} =: J_{\vec{F}}(\vec{z}_0).$$

Jacobian matrix

A partial converse is given by the

$\boxed{\text{Proposition}}$ The 4 partials exist and are continuous on $U \Rightarrow (*)$ holds (with $A = J_{\vec{F}}(\vec{z}_0)$) for all $\vec{z}_0 \in U$.

Proof: For simplicity assume $\vec{z}_0 = \vec{0} = \vec{F}(\vec{x}_0)$.

$$\Big\|\vec{F}(\vec{h}) - J_{\vec F}(\vec 0)\cdot \underbrace{\vec{h}}_{\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}}\Big\| \underset{\Delta\text{ inequality}}{\le} \text{(i)}\ \left\|\vec{F}\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} - \vec{F}\begin{pmatrix}\Delta x\\ 0\end{pmatrix} - J_{\vec F}\begin{pmatrix}\Delta x\\ 0\end{pmatrix}\cdot\begin{pmatrix}0\\ \Delta y\end{pmatrix}\right\|$$

$$+\ \text{(ii)}\ \left\|\vec{F}\begin{pmatrix}\Delta x\\ 0\end{pmatrix} - \vec{F}\begin{pmatrix}0\\ 0\end{pmatrix} - J_{\vec F}\begin{pmatrix}0\\ 0\end{pmatrix}\cdot\begin{pmatrix}\Delta x\\ 0\end{pmatrix}\right\|$$

$$+\ \text{(iii)}\ \left\|J_{\vec F}\begin{pmatrix}\Delta x\\ 0\end{pmatrix}\cdot\begin{pmatrix}0\\ \Delta y\end{pmatrix} + J_{\vec F}\begin{pmatrix}0\\ 0\end{pmatrix}\cdot\begin{pmatrix}\Delta x\\ 0\end{pmatrix} - J_{\vec F}\begin{pmatrix}0\\ 0\end{pmatrix}\cdot\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}\right\|$$

Now

$$\text{(i)} = \left\|\begin{pmatrix} u\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} - u\begin{pmatrix}\Delta x\\ 0\end{pmatrix} - u_y\begin{pmatrix}\Delta x\\ 0\end{pmatrix}\Delta y \\ v\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} - v\begin{pmatrix}\Delta x\\ 0\end{pmatrix} - v_y\begin{pmatrix}\Delta x\\ 0\end{pmatrix}\Delta y \end{pmatrix}\right\|$$

$\dfrac{\text{each}}{y} \to 0$ (by definition of the partial derivatives $u_y\begin{pmatrix}\Delta x\\ 0\end{pmatrix}$, $v_y\begin{pmatrix}\Delta x\\ 0\end{pmatrix}$)

since $\dfrac{|\Delta y|}{\|\vec h\|} \le 1$, $\dfrac{\text{(i)}}{\|\vec h\|} \to 0$ with $\|\vec h\|$.

A similar argument yields $\dfrac{\text{(ii)}}{\|\vec h\|} \to 0$. Finally,

$$\text{(iii)} = \left\|\begin{pmatrix} u_y\begin{pmatrix}\Delta x\\ 0\end{pmatrix}\Delta y + \cancel{u_x\begin{pmatrix}0\\ 0\end{pmatrix}\Delta x} - \left[\cancel{u_x\begin{pmatrix}0\\ 0\end{pmatrix}\Delta x} + u_y\begin{pmatrix}0\\ 0\end{pmatrix}\Delta y\right] \\ v_y\begin{pmatrix}\Delta x\\ 0\end{pmatrix}\Delta y + \cancel{v_x\begin{pmatrix}0\\ 0\end{pmatrix}\Delta x} - \left[\cancel{v_x\begin{pmatrix}0\\ 0\end{pmatrix}\Delta x} + v_y\begin{pmatrix}0\\ 0\end{pmatrix}\Delta y\right] \end{pmatrix}\right\|$$

$\Downarrow$

$$\frac{\text{(iii)}}{\|\vec h\|} = \underbrace{\frac{|y|}{\|\vec h\|}}_{\le 1} \cdot \underbrace{\left\|\begin{pmatrix} u_y\begin{pmatrix}\Delta x\\ 0\end{pmatrix} - u_y\begin{pmatrix}0\\ 0\end{pmatrix} \\ v_y\begin{pmatrix}\Delta x\\ 0\end{pmatrix} - v_y\begin{pmatrix}0\\ 0\end{pmatrix}\end{pmatrix}\right\|}_{\to 0} \to 0$$

by the continuity of u_y and v_y. $\square$

Now call $f: U \to \mathbb{C}$ "complex differentiable" at $z_0 \in U$, iff $\exists\ \alpha \in \mathbb{C}$ such that

$$\boxed{\lim_{h \to 0} \frac{|f(z_0+h) - f(z_0) - \alpha h|}{|h|} = 0} \;;$$

denote $\alpha =: f'(z_0) = \left.\frac{df}{dz}\right|_{z_0}$.

$\boxed{\text{Definition}}^{\dagger}$ <u>f is holomorphic on $U \iff$</u>

<u>f is complex differentiable at every point of U.</u>

<u>Comparing $\vec{F}$ & f:</u> Writing $\vec{F} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$, $f = u(x,y) + iv(x,y)$,

$$\left.\begin{array}{l} \bullet\ f \text{ is continuous (at } z_0) \iff \lim\limits_{h\to 0} f(z_0+h) = f(z_0) \\ \bullet\ \vec{F} \text{ is continuous (at } \vec{z}_0) \iff \lim\limits_{\vec{h}\to\vec{0}} \vec{F}(\vec{z}_0+\vec{h}) = \vec{F}(\vec{z}_0) \end{array}\right\} \text{same thing.}$$

$\dagger$ WARNING I don't consider this to be a correct definition of (complex) "analytic", even though they will turn out to be the same. This is my only serious terminological difference with Ahlfors.

(Moreover, it's clear that in both cases differentiability $\Rightarrow$ continuity.)

Let's write out complex differentiability in matrix/vector form: with $\alpha = a+ib$, $h = \Delta x + i\Delta y$, $f = u+iv$, we have

$$(*) \quad \lim_{h\to 0} \frac{|f(z_0+h) - f(z_0) - \alpha h|}{|h|} = \lim_{\vec{h}\to\vec{0}} \frac{\left\| \vec{F}(\vec{z}_0+\vec{h}) - \vec{F}(\vec{z}_0) - \begin{pmatrix} \mathrm{Re}(\alpha h) \\ \mathrm{Im}(\alpha h) \end{pmatrix} \right\|}{\|\vec{h}\|}$$

$$= \lim_{\vec{h}\to\vec{0}} \frac{\left\| \vec{F}(\vec{z}_0+\vec{h}) - \vec{F}(\vec{z}_0) - \overbrace{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}}^{A} \cdot \vec{h} \right\|}{\|\vec{h}\|}$$

KEY COMPUTATION:

$$\alpha h = (a+ib)(\Delta x + i\Delta y)$$
$$= (a\Delta x - b\Delta y) + i(a\Delta y + b\Delta x)$$
$$\Rightarrow \begin{pmatrix} \mathrm{Re}(\alpha h) \\ \mathrm{Im}(\alpha h) \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

The form of A suggests that complex differentiability (of f) is more "rigidifying" than real differentiability (of $\vec{F}$). The next result makes this more precise.

Theorem (a) If u_x, u_y, v_x, v_y exist and are continuous on U with

$$\boxed{u_x = v_y, \quad u_y = -v_x}$$

Cauchy-Riemann equations

then f is holomorphic on U.

(b) If f is holomorphic on U, then the 4 partials exist everywhere and satisfy the C-R equations.

Proof: (b) f $\mathbb{C}$-differentiable $\Rightarrow$ LHS $\binom{*}{*} = 0$

$\Rightarrow$ RHS $\binom{*}{*} = 0$ $\Rightarrow$ $\vec{F}$ differentiable with $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = J_{\vec{F}}$ everywhere of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$\Rightarrow$ C-R equations

(a) $\exists$ & continuity of partials $\xRightarrow[\text{Proposition}]{}$ F diff'able.

C-R equations $\Rightarrow$ $J_{\vec{F}} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}$ $\Rightarrow$ RHS $\binom{*}{*} = 0$

$\Rightarrow$ LHS $\binom{*}{*} = 0$ $\Rightarrow$ f holomorphic. $\square$

Alternate proof of (b): The definition of complex differentiability says that

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \quad \left(= f'(z_0)\right) \text{ exists.}$$

In particular, it is independent of the direction by which $h \to 0$. So we have

$$\lim_{\substack{\Delta y\to 0\\ \mathbb{R}}} \frac{f(z+i\Delta y)-f(z)}{i\Delta y} = \lim_{\substack{h\to 0\\ \mathbb{C}}} \frac{f(z+h)-f(z)}{h} = \lim_{\substack{\Delta x\to 0\\ \mathbb{R}}} \frac{f(z+\Delta x)-f(z)}{\Delta x}$$

$$\lim_{\Delta y\to 0}\left(\frac{u(x,y+\Delta y)-u(x,y)}{i\Delta y} + i\,\frac{v(x,y+\Delta y)-v(x,y)}{i\Delta y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\lim_{\Delta x\to 0}\left(\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x} + i\,\frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

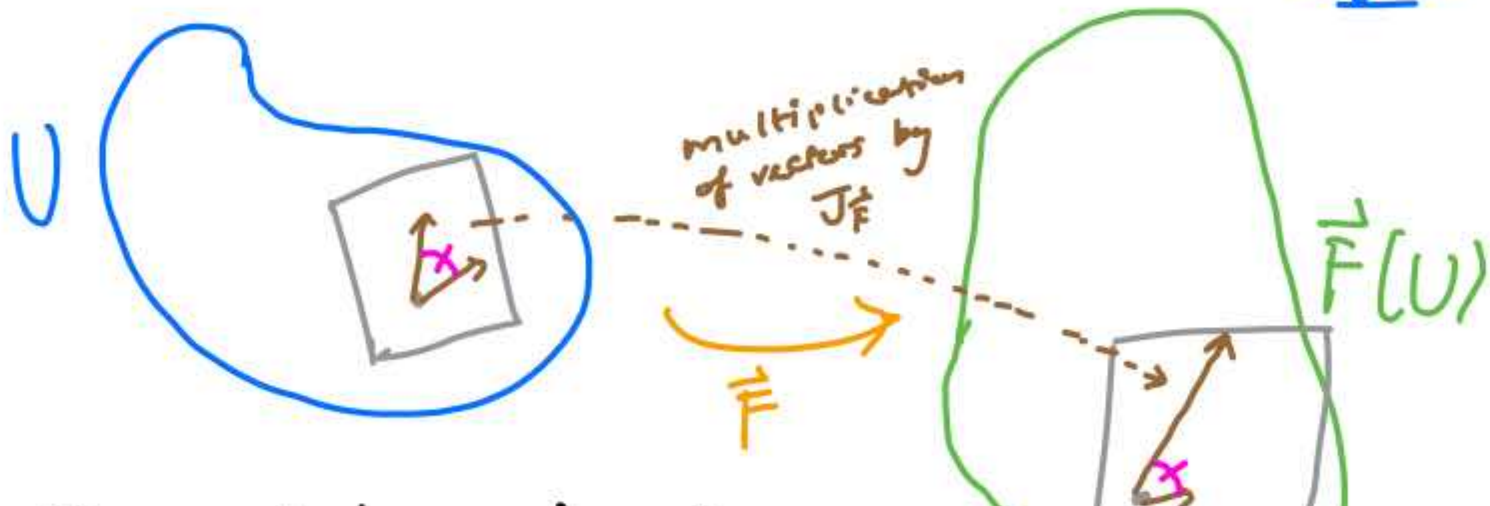
So

$$\underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{\substack{\text{real}\\ =\\ 0}} + i\underbrace{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)}_{\substack{\text{real}\\ =\\ 0}} = 0$$

, done. $\square$

Remark on $J_{\vec{F}}$: using $|\alpha| = \sqrt{a^2+b^2}$, $\varphi = \arg(\alpha)$,† the polar form $\alpha = |\alpha| e^{i\varphi}$ has a related matrix factorization

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \underbrace{\begin{pmatrix} |\alpha| & 0 \\ 0 & |\alpha| \end{pmatrix}}_{\text{dilation}} \times \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation}} \longrightarrow \text{preserves angles!}$$



$J_{\vec{F}}$ should be viewed as the infinitesimal linear transformation on vectors given by $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$; clearly this doesn't change the angles between vectors. The "integrated" form of this statement is that $\vec{F}$ (or f) doesn't change the angle between (the tangent vectors of) curves where they meet; i.e. f is CONFORMAL. ††

†† Note that the polar form expresses any complex number as a rotation $e^{i\theta}$ times a dilation r. In the HW you'll make the correspondence between matrices and complex numbers a bit more formal.

† $\cos\varphi = \dfrac{a}{\sqrt{a^2+b^2}}$, $\sin\varphi = \dfrac{b}{\sqrt{a^2+b^2}}$