Lecture 19 : Cauchy's Theorem (II)

I. Homology version of Cauchy

Recall from the end of Lect. 18 that for a closed 1-chain γ in a region $U \subset \mathbb{C}$,

$$\gamma \underset{\text{hom}}{\equiv} 0 \ (\text{in } U) \overset{\text{def.}}{\Longleftrightarrow} W(\gamma, \alpha) = 0 \ (\forall \alpha \in U^c)$$

$$\Longleftrightarrow \gamma = \partial K \ (\text{for some 2-chain } K \text{ in } U).$$

$\boxed{\text{Theorem (Cauchy v. 3.0)}}$ $f \in \mathcal{H}ol(U)$ & $\gamma \underset{\text{hom}}{\equiv} 0$ in U

$$\Longrightarrow \int_\gamma f(z)\,dz = 0.$$

Proof: $\gamma = \partial K$, where $K = \sum m_i T_i$; and we can make the T_i as small as desired, e.g. to fit $T_i \subset \underset{(\text{disk})}{D} \subset U \to \partial T_i$ homotopic to 0 in U

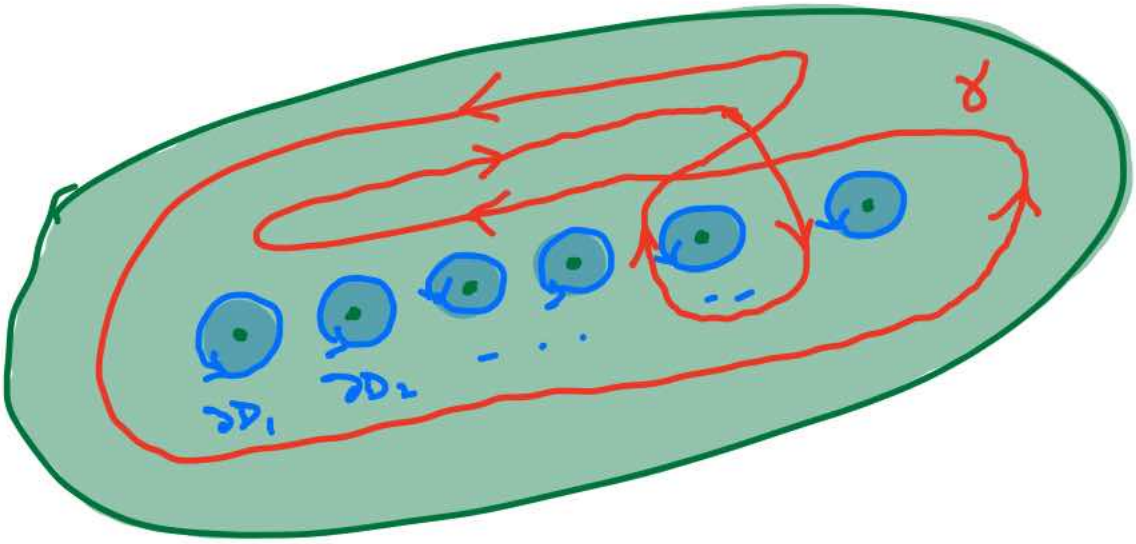
$$\Rightarrow \int_\gamma f\,dz = \int_{\sum m_i \partial T_i} f\,dz = \sum m_i \underbrace{\int_{\partial T_i} f\,dz}_{=0 \text{ by homotopy Cauchy}} = 0$$

$\square$

Now let $z_1, \ldots, z_m$ be points in U; write

$$U^* = U \setminus \{z_1, \ldots, z_m\},$$

and $D_1, \ldots, D_m$ for "small" disks around the $\{z_i\}$.



Let $\gamma \subset U^*$ be a closed 1-chain,

and assume $\gamma \underset{\text{hom}}{\equiv} 0$ ON U (but not U^*).

Claim: $\Gamma := \sum_i W(\gamma, z_i)\, \partial D_i - \gamma$ is homologous to 0 on U^*.

Proof: Let $\alpha \in (U^*)^c$.

CASE 1: $\alpha = z_j$ for some j.

$$W(\Gamma, z_j) = W\Big(\sum_i W(\gamma, z_i)\, \partial D_i - \gamma,\ z_j\Big)$$

$$= \sum_i W(\gamma, z_i)\, \underbrace{W(\partial D_i, z_j)}_{\delta_{ij}} - W(\gamma, z_j)$$

$$= W(\gamma, z_j) - W(\gamma, z_j) = 0 .$$

<u>CASE 2</u>: $\alpha \in U^c$.

$\gamma \underset{\text{hom}}{\equiv} 0$ on $U \Rightarrow W(\gamma, \alpha) = 0$

∂D_i (~ 0 on $U \Rightarrow$) $\underset{\text{hom}}{\equiv} 0 \Longrightarrow W(\partial D_i, \alpha) = 0$

$$\Longrightarrow W(\Gamma, \alpha) = 0 . \qquad \square$$

So <u>on U^*</u>, $\gamma \underset{\text{hom}}{\equiv} \sum_i W(\gamma, z_i)\, \partial D_i$.

Hence, given a function $f \in \mathcal{Hol}(U^*)$, we have

$$(\#) \qquad \int_\gamma f(z)\, dz = \sum_i W(\gamma, z_i) \int_{\partial D_i} f(z)\, dz .$$

We'll use this now to get the Cauchy integral formula.

II. Cauchy integral formula

Continuing from above, we have

Theorem Given: U open, γ closed C^0 path which is $\underset{\text{hom}}{\equiv} 0$ in U, $f \in Hol(U)$, $z_0 \in U \setminus |\gamma|$.

Then $\int_\gamma \frac{f(z)}{z - z_0} dz = 2\pi i \, W(\gamma, z_0) \cdot f(z_0)$.

Proof: f holo. on $U \implies$ analytic at z_0.

$\therefore$ on some small disk about z_0, containing a smaller disk around whose boundary we'll integrate,

$$f(z) = \sum_{k \geq 0} a_k (z - z_0)^k .$$

We CAN'T substitute this in $\int_\gamma \frac{f(z)}{z - z_0} dz$, as γ may lie outside the disk of convergence. But we CAN write

$$\int_\gamma \underbrace{\frac{f(z)}{z-z_0}}_{\in\, Hol(U\setminus\{z_0\})} dz \overset{\text{by } (\#)}{=} W(\gamma, z_0)\int_{\partial D} \frac{f(z)}{z-z_0}\, dz$$

$$= W(\gamma, z_0) \sum_{k\geq 0} a_k \underbrace{\int_{\partial D} (z-z_0)^{k-1}\, dz}_{=0 \text{ for } k\geq 1}$$

$$= W(\gamma, z_0)\, a_0 \int_{\partial D} dz/(z-z_0)$$

$$= 2\pi i\, W(\gamma, z_0) \underbrace{a_0}_{= f(z_0)}$$

$\square$

A more general result is:

$f \in Hol(U)$, γ avoids $z_1, \dots, z_m$

$$\Rightarrow \int_\gamma \frac{f(z)\, dz}{(z-z_1)\cdots(z-z_m)} = 2\pi i \sum_j W(\gamma, z_j) f(z_j).$$

Right?

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NO!!! What's wrong?

Turns out, $f(z_j)$ on r.h.s. needs to be replaced by

$$\frac{f(z_j)}{(z_j - z_1)\cdots\widehat{(z_j - z_j)}\cdots(z_j - z_m)}.$$

We'll address this later in greater generality (in the context of the residue theorem).

<u>HW</u>: You'll play w/ simpler case of multiple poles, the general case of which is (for $\gamma \underset{\text{hom}}{\simeq} 0$ in U)

$$(*) \quad \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_i W(\gamma, z_i)\,\mathrm{ord}_{z_i}(f),$$

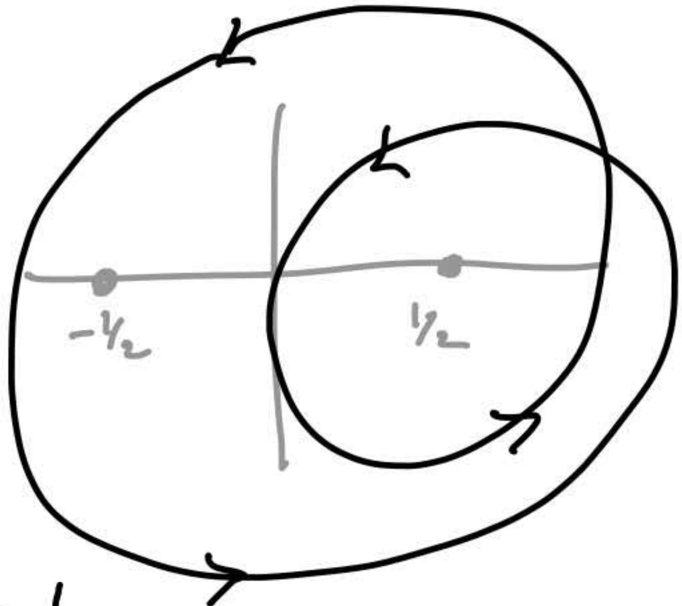
where z_i runs through the zeroes of f in U.

<u>Idea</u>: $\int_\gamma$ "$d\log(f(z))$" counts the number of times (i) $\cancel{\arg}(f(z))$ picks up $2\pi(i)$ if we don't make a "cut", which corresponds to the number of times we go around the zeroes of f (as z follows γ). So we need to multiply

($W(\gamma,\cdot)=$) # of times around by the multiplicity ($=\operatorname{ord}(f)$) of the zero, to get how many times $f(z)$ goes around 0. Hence, (*) is called the "argument principle", which we'll also return to later.

Simple Application//

$$f(z)=\frac{e^{\pi i z}}{z^2-1/4}$$

$\gamma=$ 

$W(\gamma,\tfrac12)=2$, $W(\gamma,-\tfrac12)=1$.

$$f(z)=\frac{e^{\pi i z}}{(z-\frac12)(z+\frac12)}=\frac{e^{\pi i z}}{z-\frac12}-\frac{e^{\pi i z}}{z+\frac12}$$

$$\int_\gamma f(z)=\int_\gamma\frac{e^{\pi i z}}{z-\frac12}dz-\int_\gamma\frac{e^{\pi i z}}{z+\frac12}$$

$$=2\pi i\underbrace{W(\gamma,\tfrac12)}_{2}\underbrace{e^{\pi i\frac12}}_{i}-2\pi i\underbrace{W(\gamma,-\tfrac12)}_{1}\underbrace{e^{\pi i(-\frac12)}}_{-i}$$

$$=-2\pi\cdot 2-2\pi\cdot 1$$

$$=-6\pi.$$ //

III. Local (a survey) mapping theorems

Let $U, V \subseteq \mathbb{C}$ be open sets,

$$f : U \to V \quad \text{analytic};$$

then

- f is (globally analytically) invertible $\overset{\text{Def.}}{\Longleftrightarrow}$

  $\exists\ g : V \to U$ analytic s.t.

  $g \circ f = \mathrm{Id}_U$, $f \circ g = \mathrm{Id}_V$

  $\overset{\boxed{\text{Theorem}}}{\Longleftrightarrow}$ f injective and surjective.

  ($\Rightarrow$ is clear)

Here we are talking about composition inverse, not multiplicative inverse.

- f is locally (analytically) invertible at $z_0 \in U$ $\Longleftrightarrow$

  $\exists\ \underset{z_0 \in}{U_0} \underset{\text{open}}{\subseteq} U$ s.t. $f|_{U_0} : U_0 \to f(U_0)$ is invertible

(Inverse mapping Theorem) $\boxed{\text{IMT}}$

$$\Longleftrightarrow \quad f'(z_0) \neq 0$$

$$\Longleftrightarrow \quad f(z) \underset{\text{near } z_0}{=} a_0 + \underset{\substack{\neq \\ 0}}{a_1} (z - z_0) \big(\underbrace{1 + h(z - z_0)}_{\text{power series in } z - z_0} \big)$$

True or False:

$$\left\{ \begin{matrix} f \text{ locally invertible} \\ \text{at every } z_0 \in U \end{matrix} \right\} \overset{?}{\Longleftrightarrow} \left\{ f \text{ (globally) invertible} \right\}$$

$\vdots$

False. The implication "$\Longrightarrow$" doesn't hold: e.g.

- $f(z) = z^m$, $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$
- $f(z) = \exp(z)$, $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$.

Recall $U \subset \mathbb{C}$ is (topologically) connected if we cannot write $U = V \sqcup W$ with both V & W open (hence also closed in U).

Let $f : U \to \mathbb{C}$ be analytic, and

U connected open (i.e. a region):

- f open $\overset{\text{Def.}}{\Longleftrightarrow}$ f takes open sets (in U) to open sets (in $\mathbb{C}$)

(Open mapping Theorem) $\overset{\text{OMT}}{\Longleftrightarrow}$ f not constant on U

$\Longleftrightarrow$ some $f^{(m)}(z_0) \neq 0$ $(\forall z_0 \in U)$, $m \geq 1$

$$\Longleftrightarrow f(z) \underset{\text{near } z_0}{=\!=} a_0 + \underset{\neq 0}{a_m}(z - z_0)^m (1 + h(z - z_0)) \quad (\forall z_0 \in U)$$

Remark: If $w = f(z)$ in the above, then taking

local holo. coords.

$$\tilde{w} = w - a_0 \quad \text{and}$$

$$\tilde{z} = (z - z_0)\underbrace{(1 + h(z - z_0))^{1/m}}_{\text{well-defined analytic}}$$

the mapping takes the form $\tilde{w} = \tilde{z}^m$.

IV. The Maximum Modulus Principle

Continuing to assume that f is analytic on a connected[†] open set U, we have the

Theorem (MMP)

Suppose that for some $z_0 \in U$, $|f(z_0)| \geq |f(z)|$ (OR $\mathrm{Re}\, f(z_0) \geq \mathrm{Re}\, f(z)$ OR $\mathrm{Im}\, f(z_0) \geq \mathrm{Im}\, f(z)$) for every $z \in U$. Then f is constant.

Proof 1: Assume f not constant; then f is open. Hence for $D = D(z_0, \epsilon) \subset U$, $f(D) \subseteq \mathbb{C}$ is open. Consider

$f(z_0)$

$$w := \underbrace{f(z_0)}_{} + \begin{cases} (\epsilon/2) \exp(i \arg f(z_0)) \\ \epsilon/2 \\ i\epsilon/2 \end{cases} \in f(D),$$

which has greater $\begin{cases} |\cdot| \\ \mathrm{Re} \\ \mathrm{Im} \end{cases}$ than $f(z_0)$, yet $= f(z_1)$ for some $z_1 \in D \subset U$. Contradiction. $\square$

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† otherwise there's an obvious counterexample!

Proof 2: $\leftarrow$ for 1.1

For any suff. small $\epsilon > 0$,

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z - z_0}\,dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + \epsilon e^{i\theta})}{z_0 + \epsilon e^{i\theta} - z_0}\, i\epsilon e^{i\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \epsilon e^{i\theta})\, d\theta$$

$$\Rightarrow |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \epsilon e^{i\theta})|\, d\theta,$$

and $|f(z_0)|$ maximal

$$\Rightarrow |f(z_0 + \epsilon e^{i\theta})| = |f(z_0)| \quad (\forall \theta, \epsilon)$$

$\Rightarrow f$ constant on D, hence on U. $\square$