Lecture 18: Liouville's Theorem; Homology classes

I. Entire functions

An entire function is just an element of $\mathcal{H}ol(\mathbb{C})$.

The classic result here is

Theorem (Liouville) f entire and bounded (i.e. $|f(z)| \le C \ \forall z \in \mathbb{C}$) $\Rightarrow$ f constant.

Corollary 1 If a function is entire and nonconstant, then it is unbounded.

Examples// cos, sin, exp. //

Corollary 2 Any nonconstant polynomial $P(z) = a_n z^n + \cdots + a_0$ has a root in $\mathbb{C}$.

$(n>0,\ a_n \neq 0)$

Proof of Cor 2: Assume P has no root; then $\frac{1}{P}$ is entire. Writing

$$P(z) = a_n z^n \left(1 + \frac{b_1}{z} + \cdots + \frac{b_n}{z^n}\right),$$

we find that for any $M > 0$ $\exists R \in \mathbb{R}_+$ s.t. $|z| > R \Rightarrow |P(z)| > M$. Hence $\frac{1}{P(z)} \to 0$ as $|z| \to \infty$, and defining $\frac{1}{P}(\infty) := 0$ therefore makes $\frac{1}{P}$ continuous on $\hat{\mathbb{C}}$. Now compactness of $\hat{\mathbb{C}} \Rightarrow \frac{1}{P}$ is bounded $\xRightarrow{\text{Liouville}}$ $\frac{1}{P}$ constant $\Rightarrow P$ constant. $\square$

Proof of Liouville: f entire $\xRightarrow{\text{absence of obstructions (see Lecture 17)}}$ power series for f at 0 converges on all of $\mathbb{C}$ $(r = \infty)$.

So if all $a_1, a_2, \ldots = 0$ in this expansion, then $f (= a_0)$ is constant.

Let $\gamma_R = \partial D_R$, $R > 0$ arbitrary. Recall that the Cauchy integral formula yields a power series expansion about any point in D_R; in particular, about 0 we have

$$f(z) = \sum_{n \geq 0} \underbrace{\left(\frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)\,dw}{w^{k+1}} \right)}_{a_k \left(= \frac{f^{(k)}(0)}{k!} \right)} z^k .$$

Then

$$\begin{aligned} |a_k| &= \frac{1}{2\pi} \left| \int_{\gamma_R} \frac{f(w)\,dw}{w^{k+1}} \right| \\ &\leq \frac{1}{2\pi} \cdot L(\gamma_R) \cdot \left\| \frac{f(w)}{w^{k+1}} \right\|_{\gamma_R} \\ &\leq \frac{1}{2\pi} \cdot 2\pi R \cdot \frac{C}{R^{k+1}} \\ &= \frac{C}{R^k} \xrightarrow[R \to \infty]{} 0 \quad \text{for } k \geq 1. \end{aligned}$$

$\square$

Generalization: f entire with polynomial bound, i.e. $|f(z)| \leq C |z|^m$ $\forall |z| (\geq R_0)$ suff. large

$\Longrightarrow$ f is a polynomial of degree $\leq m$. HW

Remark: If you want to get rid of "for all $|z_0| \geq R_0$", then make the bound instead $|f(z)| \leq C_0 + C_1 |z|^m$ ($\forall z \in \mathbb{C}$).

This implies the above bound with $C = 1 + C_1$, $R_0 = C_0^{1/m}$:

$$|z| \ge C_0^{1/m} \Rightarrow |z|^m \ge C_0 \Rightarrow \underbrace{|z|^m + C_1|z|^m}_{(1+C_1)|z|} \ge C_0 + C_1|z|^m .$$

Idea of proof ($m = 1$; HW = general case):

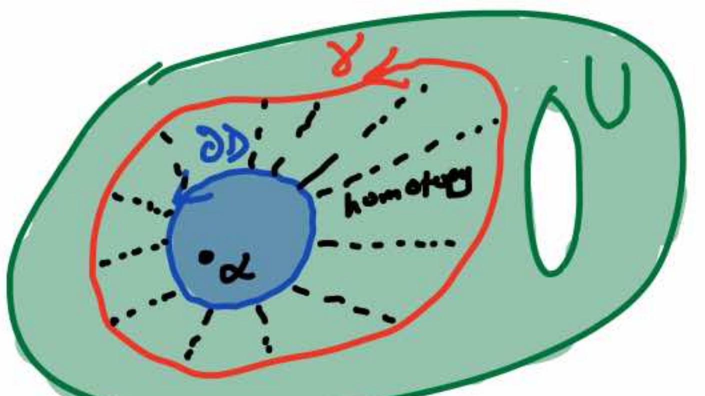
- $|f(z)| \le C|z|$, for $|z| \ge R_0$
- so $|a_k| \le \dfrac{2\pi R}{2\pi} \left\| \dfrac{f(w)}{w^{k+1}} \right\|_{\gamma_{R}} \le R \cdot \dfrac{C \cdot R}{R^{k+1}} = \dfrac{C}{R^{k-1}} \xrightarrow[(R \to \infty)]{} 0$

  $R \ge R_0$

  if $k \ge 2$ $\square$

II. Homotopy classes of paths

The point of Cauchy's integral formula is to compute integrals, and we want a stronger version than the "homotopy form". Let's recall what we know: given



- U open
- $\alpha \in \underbrace{D}_{\text{disk}} \subset U$
- $f \in \mathcal{H}ol(U)$
- $\gamma \underset{\text{homotopic to}}{\simeq} \partial D$ in $U \setminus \{\alpha\}$

Then: (†) $$\boxed{f(\alpha) \underset{\int \text{ formula}}{\overset{\text{Cauchy}}{=\!=\!=}} \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z-\alpha}\,dz \underset{\text{Cauchy theorem}}{\overset{\text{homotopy}}{=\!=\!=}} \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-\alpha}\,dz}$$

More generally if $\gamma \underset{U\setminus\{\alpha\}}{\sim} \eta \subset D$ with $W(\eta,\alpha) = n$ then $n f(\alpha) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-\alpha}\,dz$, but this is the problem — having to be <u>homotopic</u> to something in the disk. If this is not true for our given γ, might something like (†) hold anyway? We are heading toward a <u>homology</u> version of Cauchy, which is stronger because it takes far less for closed paths to be homologous than for them to be homotopic. So let's review these competing "global topological" notions.

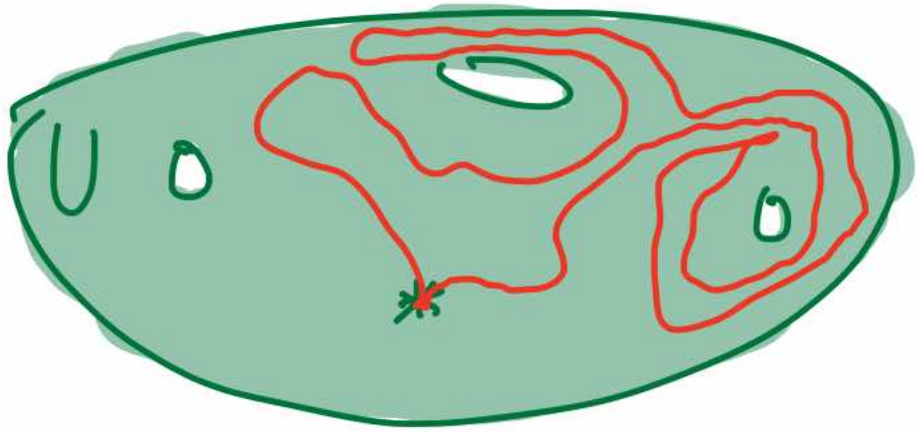
Define the "homotopy" or "fundamental" group of U — first by just specifying the underlying set:

$$\pi_1(U,*) := \frac{\{\text{closed } C^0 \text{ paths starting \& ending at } *\}}{\text{homotopy equivalence "}\sim\text{"}},$$

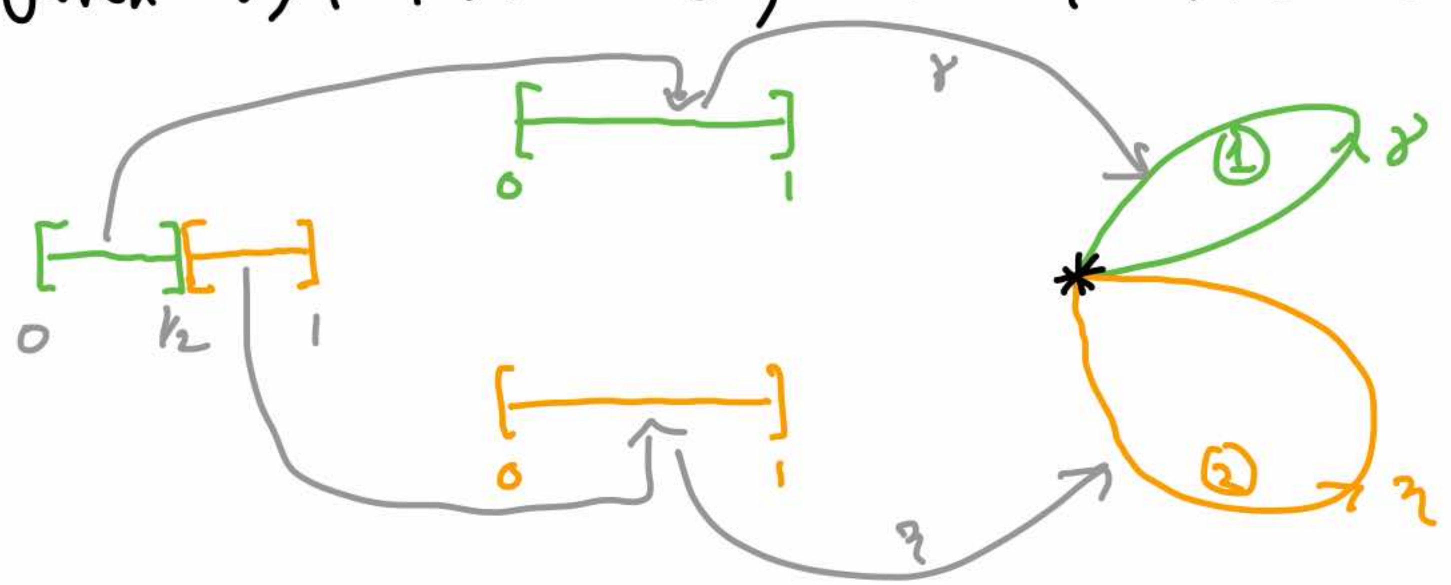
i.e. γ and η are the "same element" of π_1 $\Longleftrightarrow$ $\gamma \sim \eta$.

A path γ is the "trivial element" $\{*\}$ in π_1 $\Longleftrightarrow$ $\exists$ C^0 $\Psi : [0,1]\times[0,1] \to U$ with $\Psi_0(t) = \gamma(t)$, $\Psi_1(t) = \{*\}$ $(\forall t)$.

Intuitively, this means (viewing γ as a "rope") you can stand at "$*$", grip 2 ends of the rope and pull to yourself without the rope passing through the holes:

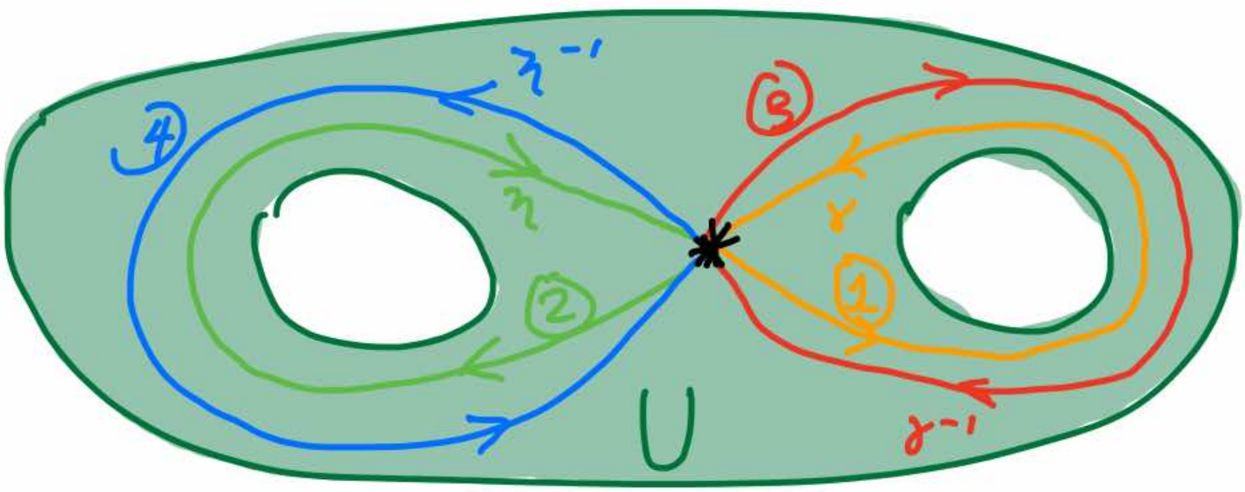


Given $\gamma, \eta : [0,1] \to U$, define $\eta \cdot \gamma : [0,1] \to U$:



This makes π_1 into a group, with identity $\{*\}$ (constant path) and "γ^{-1}" just γ traversed backwards. (Why does this work?) However, in general we'll have $\gamma \cdot \eta \not\simeq \eta \cdot \gamma$, or equivalently $\underbrace{\eta^{-1} \cdot \gamma^{-1} \cdot \eta \cdot \gamma}_{\text{"commutator" of } \eta \,\&\, \gamma} \not\simeq \{*\}$.

Crucial here is that we aren't allowed to subdivide the path and cancel pieces — more like tying the ends of 4 strings together and trying to pull the whole thing towards you (and you have beginning of γ & end of η^{-1} in your hands). This



is an example of a nontrivial commutator, i.e. one you cannot pull to $\{*\}$. Accordingly, homotopy Cauchy can't tell us anything about the integral over $\eta^{-1}\cdot\gamma^{-1}\cdot\eta\cdot\gamma$....

III. Homology classes of chains

..... which is, of course, completely ridiculous: if you integrate over $\eta^{-1}\cdot\gamma^{-1}\cdot\eta\cdot\gamma$ (regardless of the integrand $f\in \mathcal{H}ol(U)$), you get 0 by cancellation!!

Define for a closed path $\Gamma\subset U$

$$\Gamma \underset{\text{hom}}{\equiv} 0 \overset{\text{def.}}{\iff} 0 = W(\Gamma,\alpha) := \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-\alpha} \quad \forall \alpha \in \mathbb{C}\setminus U.$$

"is homologous to 0 on U"

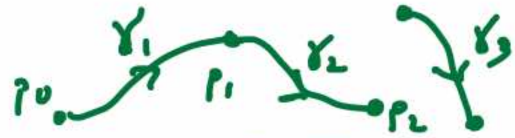
So: $\eta^{-1}\cdot\gamma^{-1}\cdot\eta\cdot\gamma \underset{\text{hom}}{\equiv} 0$.

The reason I wrote "chains", is that I want you to think of homology in terms of subdividable objects, in contrast to homotopy. Why? Because integrals can be subdivided by definition (Riemann), and homology is defined† in terms of winding numbers (which are integrals).

Definition On $U \subset \mathbb{C}$, we define

- 0-chain := formal sum of points in U (= complex #'s) with $\mathbb{Z}$-coefficients
- 1-chain := formal sum of paths in U with $\mathbb{Z}$-coeffs.

$$\sum m_i \cdot \gamma_i$$



  – can define "boundary", e.g. $\partial\gamma_2 = p_2 - p_1$
  $\partial(\gamma_1+\gamma_2) = p_2 - p_0$ 0-chains

  – each "piece" is parametrized but (except for the direction) we forget this.

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† in our ad hoc definition above; there is a better, more general one, to be given in a moment.

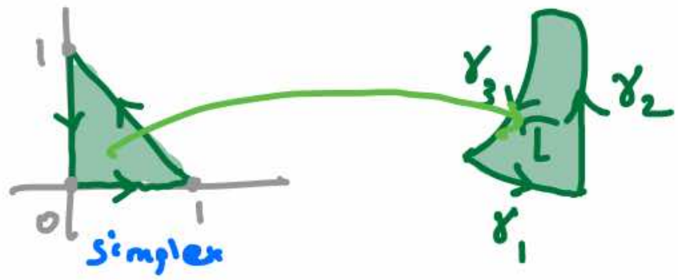
• 2-chain := formal sum of curvy triangles (in U) w/ $\mathbb{Z}$-coeffs.

$$\sum_i m_i \cdot \tau_i$$

— boundary:

$$\partial\tau = \gamma_1+\gamma_2+\gamma_3$$

1-chain, image of boundary of simplex



— can get any shape region by subdividing into curvy triangles (triangulating)

• We write sums of chains with $+$, not $\cdot$, whether or not they are end-to-end; they commute by definition.

Also,

 and 

are identified.

• A chain Γ is closed $\iff \partial\Gamma = 0$. Define the first homology group of U:

$$H_1(U) := \frac{\{\text{closed 1-chains on } U\}}{\underbrace{\partial\{\text{2-chains on } U\}}_{\text{boundaries of finite sums (w/multiplicity) of } \Delta\text{'s.}}}$$

element in here determined by γ is denoted $[\gamma]$ ("homology class")

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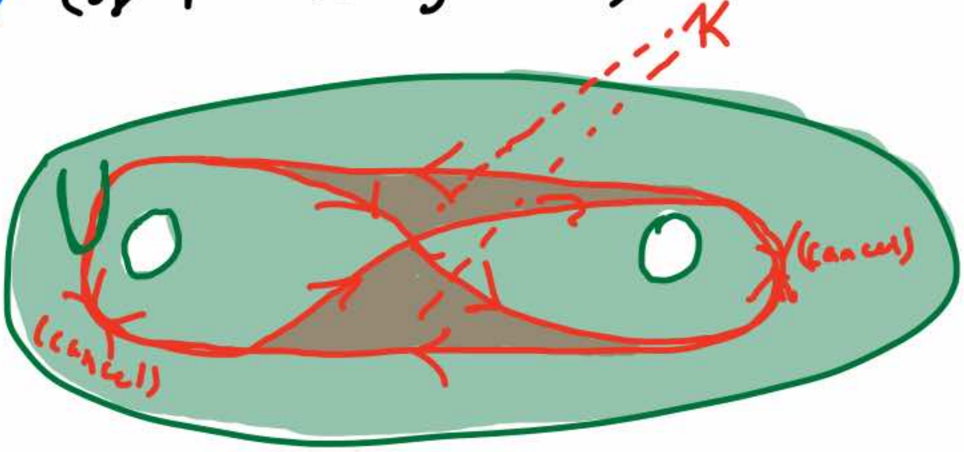
Cauchy's theorem will just say that integrals of holomorphic functions on U, are well-defined on homology classes; that is, " $\int_{[\gamma]} f\,dz$ " makes sense.

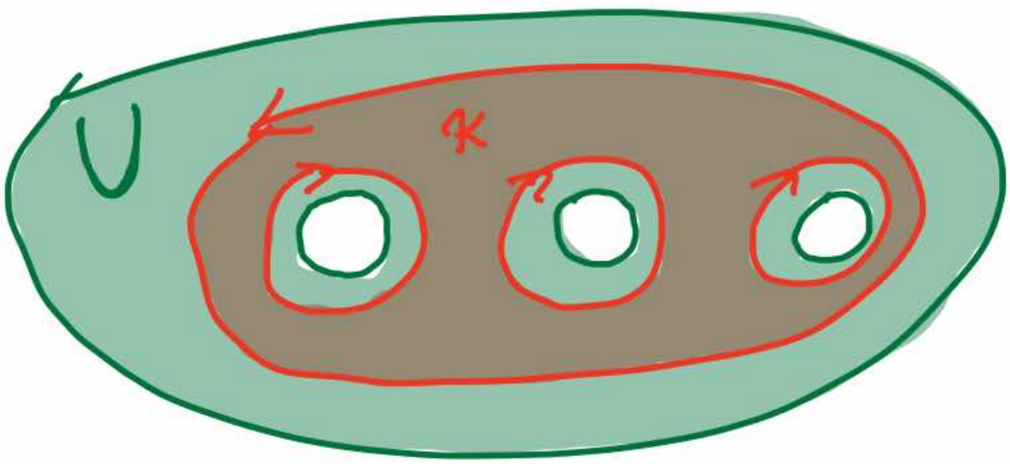
The assertion in the background is that the two definitions of "homologous" are the same:

$$\gamma \underset{\text{hom}}{\equiv} 0 \iff \underset{(\forall \alpha \in U^c)}{W(\gamma,\alpha)=0} \overset{(*)}{\iff} \underbrace{\gamma = \partial K,\ K \text{ a 2-chain in } U}_{\text{good for drawing pictures}}$$

Examples // (of Γ homologous to 0)

① 

(similar to commutator example)

② 

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Now, $\gamma \sim \{*\}$ (in U) $\iff$ path "contractible" to $*$ (in U)

$$\Longrightarrow W(\gamma,\alpha) = 0 \quad \forall \alpha \in U^c.$$

$\gamma \sim * \Rightarrow \int_\gamma \frac{dz}{z-\alpha} = 0$ by homotopy Cauchy, since $\alpha \notin U \Rightarrow \frac{1}{z-\alpha} \in Hol(U)$

So: γ "homotopic to 0" $\Rightarrow$ γ homologous to 0

but NOT vice versa.

Hurewicz homomorphism: $\pi_1(U) \to H_1(U)$

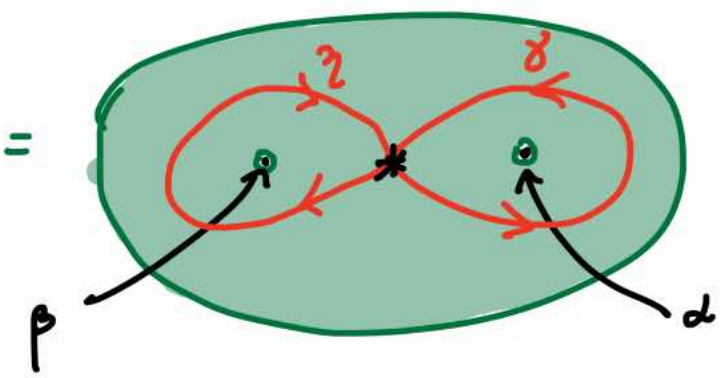
$$\gamma \longmapsto [\gamma]$$

$$\eta\cdot\gamma \longmapsto [\eta]+[\gamma]$$

$$\eta^{-1}\cdot\gamma^{-1}\cdot\eta\cdot\gamma \longmapsto 0$$

In fact, $H_1 \cong \pi_1 / \underbrace{[\pi_1,\pi_1]}_{\text{commutator subgroup, generated by all commutators}}$

Example//

U = 

$\pi_1(U,*) = \mathcal{F}\{\gamma,\eta\}$ (free group): elements are the words $\gamma^{a_1}\eta^{b_1}\gamma^{a_2}\eta^{b_2}\cdots\eta^{b_n} = \Gamma$.

$H_1(U) = \mathbb{Z} \oplus \mathbb{Z}$, the free abelian group on $[\gamma]$ & $[\eta]$.

The Hurewicz map sends $\Gamma \mapsto \underbrace{\left(\sum_{i=1}^{n} a_i\right)}_{a}[\gamma] + \underbrace{\left(\sum_{i=1}^{n} b_i\right)}_{b}[\eta] = [\Gamma]$.

Note that $a = W(\Gamma, \alpha)$, $b = W(\Gamma, \beta)$. //

Before explaining (*) above, I recall properties of the winding #.

(i) it's an integral. For all $\int$'s, $\int_{\gamma+\eta} = \int_\gamma + \int_\eta$.

(This is why homology works so well with integration.)

So $W(\gamma, \alpha) + W(\eta, \alpha) = W(\gamma+\eta, \alpha)$ for closed η, γ.

(ii) it's an integer, and it is constant as α varies in a connected component of $\mathbb{C} \setminus |\gamma|$.

Proof of (*):

① $\gamma = \partial \tau \Rightarrow W(\gamma, \alpha) = 0$ for any $\alpha \notin U$:

$\gamma = \sum m_i \cdot \partial \tau_i$, $\tau_i \subset U$. By triangulating, make $\{\tau_i\}$ as small as needed: make them fit in a circle of radius $= d(\gamma, U^c) \Rightarrow \partial \tau_i$ homotopic to 0.

(disk is convex $\Rightarrow$ simply connected)

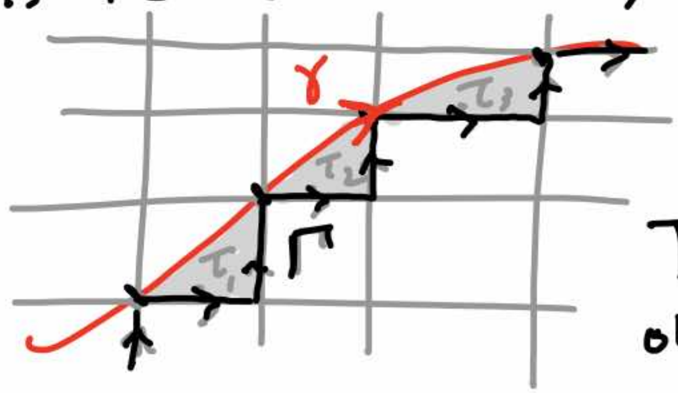
Hence,

$$\int_\gamma f\,dz = \sum m_i \int_{\partial \tau_i} f\,dz \underset{\text{htpy. Cauchy}}{=} 0 \text{ for } f \in \mathrm{Hol}(U),$$

$\partial\tau_i \sim 0$

and $\frac{1}{z-\alpha} \in \mathrm{Hol}(U)$ for $\alpha \notin U$.

② $W(\gamma,\alpha) = 0 \;\; \forall \alpha \notin U \Rightarrow \gamma = \partial K$: (This is the harder direction, as we must construct K.)



Partition γ so that points are closer together than $d(\gamma, U^c)$. Then "rectangularize" the path to obtain Γ. Clearly $\Gamma - \gamma = \partial \sum \tau_i$, and Γ, γ close together $\Rightarrow$

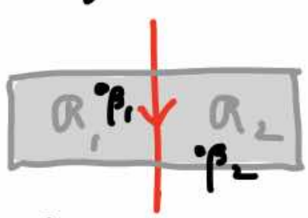
Γ, γ homotopic $\Rightarrow$ Γ, γ have same winding #'s about $\alpha \notin U$

$\Rightarrow$ can replace γ by Γ.

Now subdivide the plane into rectangles $\overline{R_i}$, using vertical & horizontal extensions of Γ, & let $\beta_i \in R_i$ be arbitrary.

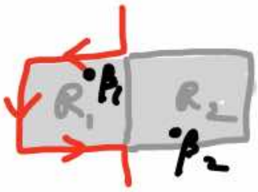
Lemma: Let η be a rectangular path, with segment between R_1 & R_2:

(w/ multiplicity, which we assume to be 1)



Then $W(\eta, \beta_1) \neq W(\eta, \beta_2)$.

Pf: "divert" the path by adding ∂R_1; then $\beta_1, \beta_2 \in$ same connected component of $\mathbb{C} \setminus |\eta + \partial R_1|$. So



$$0 = W(\eta + \partial R_1, \beta_1) - W(\eta + \partial R_1, \beta_2)$$

$$= \underbrace{\big(W(\eta, \beta_1) - W(\eta, \beta_2)\big)}_{\neq 0} + \big(\underbrace{W(\partial R_1, \beta_1)}_{= 1} - \underbrace{W(\partial R_1, \beta_2)}_{= 0}\big)$$

as desired. $\square$

Now set $m_i := W(\Gamma, \beta_i)$, $\beta_i \in R_i$ arbitrary.

$$m_i \neq 0 \Longrightarrow \beta_i \in U \Rightarrow \overline{R_i} \subseteq U.$$

(contrapositive of our hypothesis: $\alpha \notin U \Rightarrow W(\Gamma, \alpha) = 0$)

β_i arbitrary

So $K := \sum m_j \overline{R_j}$ is a 2-chain in U.

(subdivide into simplices)

Notice that for $\beta_i \in R_i$,

(4) $$W(\partial K, \beta_i) = \sum m_j \underbrace{W(\partial R_j, \beta_i)}_{\delta_{ij}} = m_i = W(\Gamma, \beta_i).$$

Claim: $\Gamma = \partial K$.

Suppose otherwise: then the closed rectangular 1-chain $\Gamma - \partial K$ is nonzero, hence has a segment. That segment divides 2 rectangles, but

(say, R_1 & R_2

$$W(\Gamma - \partial K, \beta_1) = W(\Gamma, \beta_1) - W(\partial K, \beta_1)$$

$$\underset{(4)}{=} 0 \underset{(4)}{=} W(\Gamma - \partial K, \beta_2),$$

contradicting the Lemma. $\square$

We have proved

Proposition $\underset{\text{(in } U)}{\gamma \underset{\text{hom}}{\equiv} 0} \iff \underset{\forall \alpha \notin U}{W(\gamma, \alpha) = 0} \iff \underset{\text{(in } U)}{\gamma = \partial K}$.

(You should think of K as a finitely triangulable compact subset of U.) This will lead pretty directly to "homology versions" of Cauchy's Thm. & Cauchy's Integral formula in the next lecture.