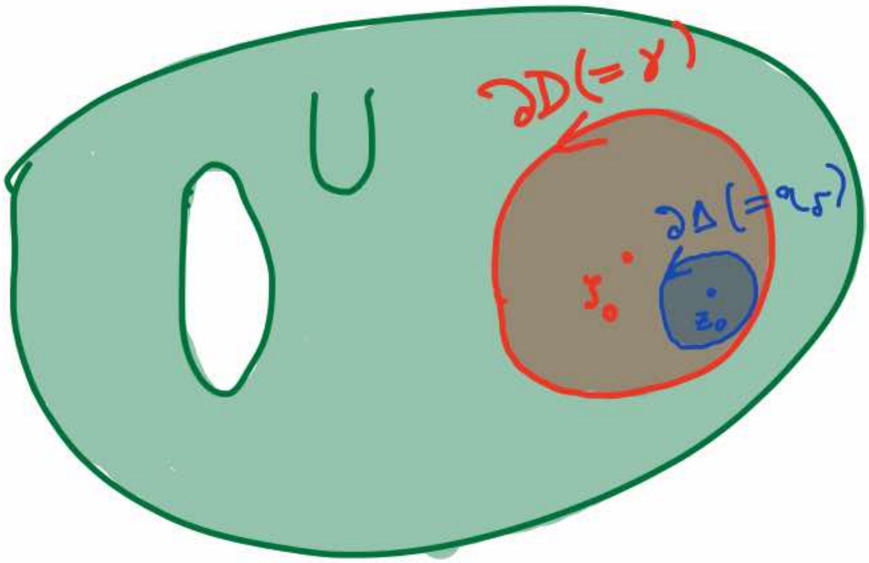
Lecture 17: The Cauchy Integral Formula

I. Reproducing holomorphic functions

Let U be a region, $D = D(\zeta_0, R)$ a disk with closure $\bar{D} \subset U$. Choose a point $z_0 \in D$, and let $\Delta = D(z_0, \delta)$ be a disk with closure $\bar{\Delta} \subset D$.



Write

$$\begin{cases} \gamma \\ \eta_\delta \end{cases} \text{ for } \begin{cases} \partial D \\ \partial \Delta \end{cases}$$

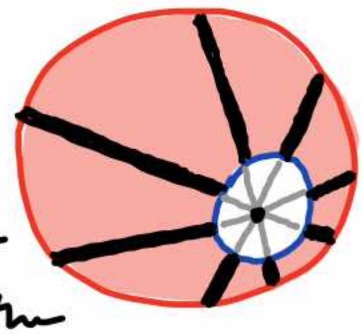
viewed as paths.

<u>Claim</u>: $\gamma \sim \eta_\delta$ on $U \setminus \{z_0\}$.

Proof: $\gamma(t) = \zeta_0 + Re^{it}$, and we can define

$$\eta_\delta(t) = z_0 + \delta \frac{\gamma(t) - z_0}{|\gamma(t) - z_0|}.$$

Check: We just need that each bold segment gives a unique point on $\partial\Delta$ corresponding to each point on ∂D and vice versa. Then the homotopy is $\Psi(s,t) := s\gamma(t) + (1-s)\eta_\delta(t)$, which maps $[0,1]\times[0,2\pi]$ onto the shaded area.



$\square$

Now $g(z) := \dfrac{f(z)-f(z_0)}{z-z_0} \in \mathrm{Hol}(U\setminus\{z_0\})$ for any $f \in \mathrm{Hol}(U)$, and then the homotopy Cauchy theorem $\Longrightarrow$

$$\int_{\eta_\delta} g\,dz = \int_\gamma g\,dz.$$

Since $\lim_{z\to z_0} g(z) = f'(z_0)$, setting $g(z_0) := f'(z_0)$ makes $g \in C^0(\overline{D})$ $\Longrightarrow$ $|g(z)| \le B \quad \forall z \in D\setminus\{z_0\}$ for some $B \in \mathbb{R}_{>0}$.

$\uparrow$ compact

So $\left|\int_\gamma g\,dz\right| = \left|\int_{\eta_\delta} g\,dz\right| \le L(\eta_\delta)\,\|g\|_{\eta_\delta} \le 2\pi\delta\cdot B \xrightarrow{\delta\to 0} 0$

$\Longrightarrow \;\; \| \;\; 0$, and $0 = \int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz = \int \frac{f(z)}{z-z_0}\,dz - \underbrace{\int \frac{f(z_0)}{z-z_0}\,dz}_{2\pi i f(z_0)}$,

proving the

$$\boxed{\textbf{Theorem (Cauchy integral formula):}\quad \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z-z_0}\,dz = f(z_0).}$$

That is, you can recover values of a holomorphic function inside a disk from its values on the boundary.

This is a consequence of the "rigidity" implied by the C-R equations / conformality.

II. Holomorphic functions are analytic

Now assume $z_0 = \zeta_0$ (= center of D); recall R = radius of D.

Thinking of z as fixed ($\in D$), $w \in \partial\overline{D}$,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)-(z-z_0)}\,dw$$

$$= \frac{1}{2\pi i}\int_\gamma \sum_{n\geq 0} \frac{f(w)\,(z-z_0)^n}{(w-z_0)^{n+1}}$$

converges uniformly since $|z-z_0| < |w-z_0|$

$$= \sum_{n\geq 0} \underbrace{\left(\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)}_{=:\, a_n} (z-z_0)^n$$

since $z-z_0$ is constant w.r.t. integration

$$= \sum_{n\geq 0} a_n (z-z_0)^n.$$

Since this is convergent (simply b/c the computation is valid), f is analytic at z_0. It's also clear (since it's convergent on D) that the radius of convergence is at least R.

[If you want an explicit check:

$$|a_n| = \frac{1}{2\pi}\left|\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw\right| \le \frac{1}{2\pi}\underbrace{L(\gamma)}_{2\pi R}\,\underbrace{\left\|\frac{f(w)}{(w-z_0)^{n+1}}\right\|_\gamma}$$

$|w-z_0| = R$ on γ, $f(w) \le C$ as γ compact

$$\le R\cdot\frac{C}{R^{n+1}}$$

$$\Rightarrow |a_n| \le \frac{C}{R^n}\ (\forall n).$$

So $\frac{1}{r} = \limsup |a_n|^{1/n} \le \lim \left(\frac{C}{R^n}\right)^{\frac{1}{n}} = \frac{1}{R} \Rightarrow r \ge R$.]

We have proved the

$\boxed{\text{Theorem}}$ $f \in \mathcal{Hol}(U) \Longleftrightarrow f \in \mathcal{An}(U)$

(we had "$\Longleftarrow$" already)

$\boxed{\text{Corollary A}}$ f $\mathbb{C}$-differentiable on U

$\Longrightarrow$ f infinitely $\mathbb{C}$-differentiable on U.

$\boxed{\text{Corollary B}}$ Let f be holomorphic on $U \subset \mathbb{C}$, $z_0 \in U$.

Then

$$\{\text{radius of convergence of [unique] power series for } f \text{ at } z_0\} \overset{①}{=}$$

$$\max\{r \mid f \text{ admits holomorphic extension to } D(z_0,r)\} \underset{②}{\geq}$$

$$(\text{distance from } z_0 \text{ to } \partial U).$$

Proof: Only thing not obvious is the "$\geq$" part of ①:

i.e., if f is holomorphic on $D(z_0, r)$ then its power series at z_0 converges there. But we now know that for $\bar{D} = \bar{D}(z_0, R) \subset D(z_0, r)$, the radius of convergence is $\geq R$ by the above argument. So radius of convergence $\geq r - \epsilon$ $\forall \epsilon > 0$, hence $\geq r$. $\square$

Example // Given $f(z) = \dfrac{C}{\prod_i (z - \alpha_i)} = \sum_{n \geq 0} A_n (z - z_0)^n$,

you would compute the $\{A_n\}$ and use root test to find r. Alternately, just use Corollary B

$\Rightarrow r = \min_i |z_0 - \alpha_i|$. //

III. Testing for holomorphicity

Morera's Theorem Let U be open, $f \in C^0(U)$, and $\int_{\partial R} f\,dz = 0$ ($\forall$ rectangles $R \subset U$).
... or circles
... or triangles
⋮

Then $f \in \mathcal{H}ol(U)$.

Proof: All $\int_{\partial R} f\,dz = 0 \xRightarrow{\text{Lect. 14}}$

$\exists$ local holo. primitive F in nbhd. of every point.

But $\mathbb{C}$-diff $\xRightarrow{\text{Cor. A}}$ ∞ - $\mathbb{C}$-diff. (for F)

$\Longrightarrow$ $f = F'$ is $\mathbb{C}$-diff'able,

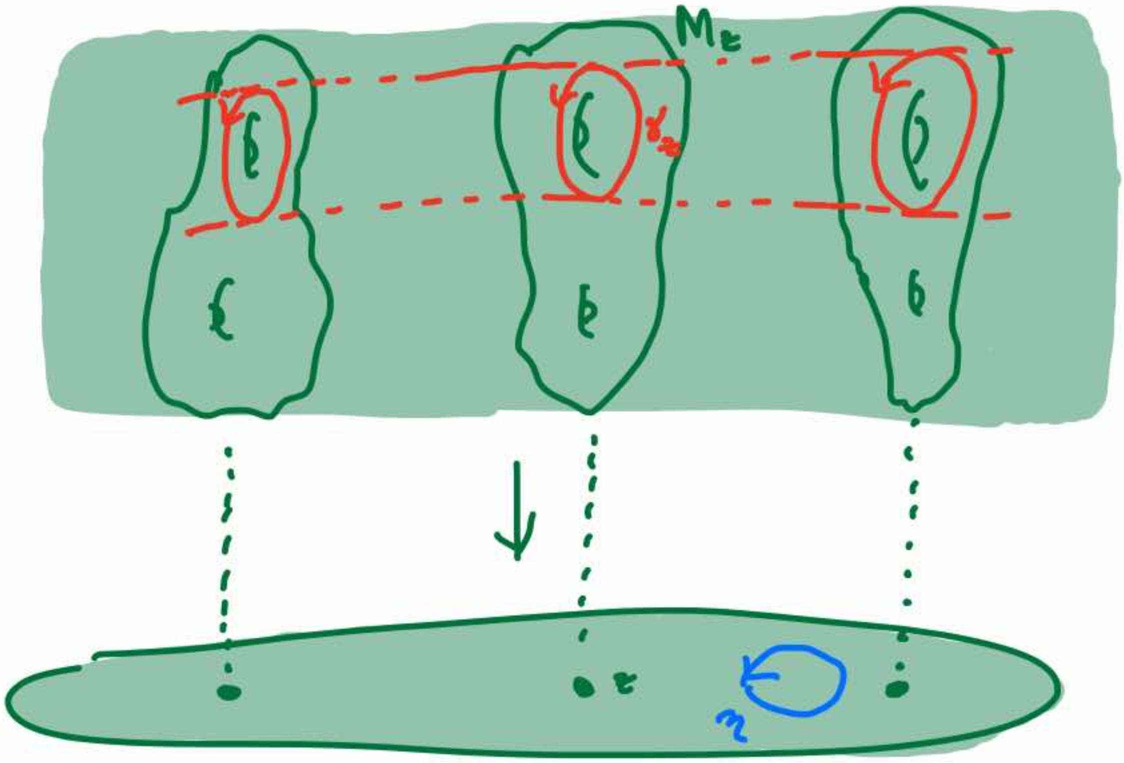
in nbhd. of every point in U &

thus in all of U. $\square$

Example // In algebraic geometry, there are various "integrals on complex manifolds which, by Morera, are shown to produce holomorphic functions as the complex structure varies.

Simplest case: a family of Riemann surfaces / alg. curves over an open set $U \subset \mathbb{C}$

$\mathcal{M} = \bigcup_{z \in U} M_z$



U

On each M_z, we can compute $f(z) = \int_{\gamma_z} \omega_z$ where ω_z is a family of holomorphic differential 1-forms. If you know that $\omega \wedge dz = \Omega$ is a holomorphic 2-form on $\mathcal{M}$, then $d\Omega = 0$.

(the 1-form restricting to ω_z on M_z)

(coordinate pulled back from U)

So then $\bigcup_{z \in \eta} \gamma_z =: \Gamma = \partial \Xi$, where $\Xi = \bigcup_{z \in (\text{area that } \eta \text{ bounds})} \gamma_z$,

and $\int_{\eta} f \, dz = \int_{\eta} \left(\int_{\gamma_z} \omega_z \right) dz = \int_{\Gamma} \Omega = \int_{\partial \Xi} \Omega$

any loop

$= \int_{\Xi} d\Omega = 0 \implies f \in \mathcal{H}ol(U)$. //

Stokes thm.

Morera thm.

IV. Generalizations of Cauchy

Now let $\gamma \subset U$ be a C^0 path (not necessarily closed) in an open set, and $g : \gamma \to \mathbb{C}$ a $\underbrace{C^0}_{\text{(can be relaxed)}}$ function defined on (the image of) γ.

$\boxed{\text{Theorem}}$ Define $f(z) := \int_\gamma \frac{g(w)}{w-z}\,dw$.

Then $f(z) \in Hol(U \setminus \gamma)$, with derivatives

$$f^{(m)}(z) = m! \int_\gamma \frac{g(w)}{(w-z)^{m+1}}\,dw.$$

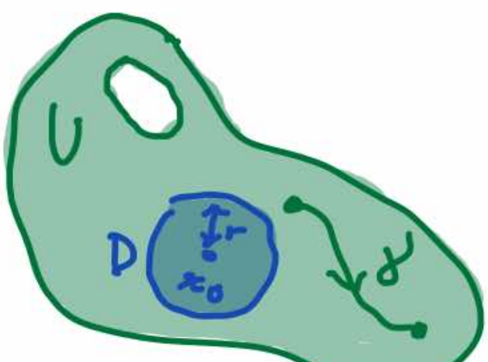
Proof: [Note: we don't have to prove $f =$ anything, hence don't have to use Cauchy $\int$ formula.]

Pick any $z_0 \in U \setminus \gamma$, and set $r := d(z_0, \partial U \cup \gamma)$.

For all $w \in \gamma$ and $z \in D(z_0, r)$,

$|z - z_0| < r \le |w - z_0|$

$$\implies \left| \frac{z - z_0}{w - w_0} \right| < 1.$$



Thus

$$f(z) := \int_\gamma \frac{g(w)}{w-z}\,dw$$

$$= \int_\gamma \left(\sum_{k\geq 0} \frac{(z-z_0)^k g(w)}{(w-z_0)^{k+1}} \right) dw$$

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)}$$

$$= \frac{1}{w-z_0} \cdot \frac{1}{1-\frac{z-z_0}{w-z_0}}$$

$$= \frac{1}{w-z_0} \sum_{k\geq 0} \left(\frac{z-z_0}{w-w_0}\right)^k ;$$

multiplying by $g(w)$, this series converges uniformly as functions on γ (because for $w \in \gamma$ the geometric series has $|\text{ratio}| < 1$)

$$= \sum_{k\geq 0} \int_\gamma \frac{(z-z_0)^k g(w)}{(w-z_0)^{k+1}} dw$$

$$= \sum_{k\geq 0} \left(\int_\gamma \frac{g(w)}{(w-z_0)^{k+1}} dw \right) (z-z_0)^k$$

$$=: \sum_{k\geq 0} a_k (z-z_0)^k .$$

Hence, $f(z)$ is analytic at z_0, with $f^{(m)}(z_0) = m!\, a_m$

$$= m! \int_\gamma \frac{g(w)}{(w-z_0)^{m+1}} dw .$$

$\square$

Remark: The Cauchy integral formula could never reproduce a continuous function which wasn't holomorphic, since by the Theorem/proof above, such integral formulae apparently always give holomorphic output. This leads one to ask whether there is a modified Cauchy integral formula for, say, C^∞ functions.

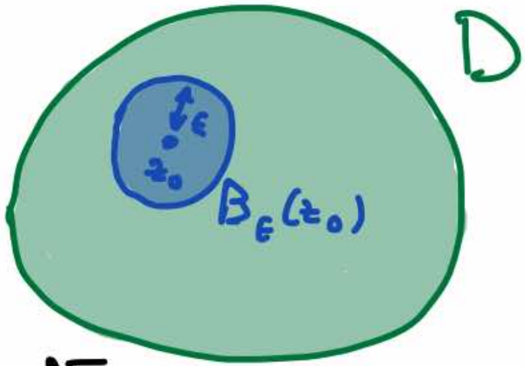
The Stokes formula says $\oint_{\partial\Gamma} g\, dz + h\, d\bar{z}$ — C^∞ fcns.

$$= \iint_\Gamma d\{g\, dz + h\, d\bar{z}\} = \iint_\Gamma \left(\frac{\partial g}{\partial \bar{z}} d\bar{z} \wedge dz + \frac{\partial h}{\partial z} dz \wedge d\bar{z} \right) = \iint_\Gamma \left(\frac{\partial h}{\partial z} - \frac{\partial g}{\partial \bar{z}} \right) dz \wedge d\bar{z}$$

Applying this to the case

$$g(z) = \frac{f(z)}{z-z_0}, \quad h(z) = 0,$$

C^∞



we get

$$\iint_D \frac{\partial f}{\partial \bar z} \frac{dz \wedge d\bar z}{z - z_0} := \lim_{\varepsilon \to 0} \iint_{D \setminus B_\varepsilon(z_0)} \frac{\partial f}{\partial \bar z} \frac{dz \wedge d\bar z}{z - z_0}$$

$$= -\lim_{\varepsilon \to 0} \int_{\partial(D \setminus B_\varepsilon(z_0))} f(z) \frac{dz}{z - z_0}$$

(note that $\partial/\partial \bar z$ of this is zero.)

$$= \lim_{\varepsilon \to 0} \oint_{C_\varepsilon(z_0)} \frac{f(z)}{z - z_0} dz - \oint_{\partial D} \frac{f(z)}{z - z_0} dz$$

i.e. $\partial B_\varepsilon(z_0)$

$$\left| \int \frac{f(z) - f(z_0)}{z - z_0} dz \right| \le 2\pi\varepsilon \left\| \frac{f(z) - f(z_0)}{z - z_0} \right\|_{C_\varepsilon} \le 2\pi \left\| f(z) - f(z_0) \right\|_{C_\varepsilon} \to 0$$

since f is C^∞

$$= 2\pi i f(z_0) - \oint_{\partial D} \frac{f(z)}{z - z_0} dz$$

$$\Longrightarrow \boxed{f(z_0) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(z)}{z - z_0} dz - \frac{1}{\pi} \iint_D \frac{\partial f}{\partial \bar z} \frac{dA}{z - z_0},}$$

where we used

$$dz \wedge d\bar z = (dx + i\,dy) \wedge (dx - i\,dy) = -2i\, dx \wedge dy = -2i\, dA$$

$$\Rightarrow \frac{1}{2\pi i} dz \wedge d\bar z = \frac{-2i}{2\pi i} dA = \frac{-1}{\pi} dA.$$

//

V. The winding number

A special case of the last theorem is where $\underline{g \equiv 1}$ and γ is $\underline{\text{closed}}$. Then

$\left(\begin{array}{l} \gamma : [a,b] \to U \\ \text{assume at least piecewise } C^1 \end{array} \right)$

$$f(z) := \int_\gamma \frac{1}{w-z}\, dw$$

$$= \int_a^b \frac{\gamma'(t)}{\gamma(t)-z}\, dt .$$

Now view z for the moment as $\underline{\text{fixed}}$, and consider the related function

$$F(s) := \int_a^s \frac{\gamma'(t)}{\gamma(t)-z}\, dt , \quad s \in [a,b] .$$

This is differentiable except at finitely many points (with the obvious derivative), and we write

$$\frac{d}{ds} \underbrace{e^{-F(s)}(\gamma(s)-z)}_{\tilde{F}(s)} = e^{-F(s)}\gamma'(s) - F'(s)\, e^{-F(s)}(\gamma(s)-z)$$

$$= e^{-F(s)}(\gamma(s)-z) \underbrace{\left(\frac{\gamma'(s)}{\gamma(s)-z} - F'(s) \right)}_{0}$$

$$= 0 .$$

So $\tilde{F}$ is $\begin{cases} \text{continuous} \\ \text{constant except (possibly) at finitely many points} \end{cases}$

$\Rightarrow \tilde{F}$ constant (on $[a,b]$).

That is, we have

$z \notin \gamma$, $0 \notin$ range of exp

$$e^{-F(s)}(\gamma(s) - z) = C \; (\neq 0)$$

$$\Rightarrow \quad \gamma(s) - z = Ce^{F(s)}$$

$$\Rightarrow \quad Ce^{F(b)} = \gamma(b) - z = \gamma(a) - z = Ce^{F(a)}$$

(γ closed)

$$\underset{C \neq 0}{\Rightarrow} \quad e^{F(b)} = e^{F(a)} = e^0 = 1$$

$$\Rightarrow \quad f(z) = F(b) = 2\pi i k, \quad \text{for some } k \in \mathbb{Z}$$

$$\Rightarrow \quad \frac{1}{2\pi i} f(z) \in \mathbb{Z}.$$

By the Theorem, $f(z) \in \mathrm{Hol}(U \setminus \gamma)$.

But holomorphic (⇒ continuous) + integral implies <u>constant on connected components</u>.

$\boxed{\text{Definition}}$ U open, $\gamma \subset U$ path of class C^1, $z \in U \setminus \gamma$. The <u>winding number of γ with respect to z</u> is

$$W(\gamma, z) := \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z}\, dw \quad (\in \mathbb{Z}).$$

We have proved:

$\boxed{\text{Proposition}}$ $W(\gamma, z)$ is integral, and constant on connected components of $U \setminus \gamma$.

Example $U = \mathbb{C}$, $z = 0$; $\gamma(t) = e^{2\pi i n t}$, $t \in [0,1]$.

$$W(\gamma, 0) = \frac{1}{2\pi i} \int_0^1 \frac{2\pi i n e^{2\pi i n t}}{e^{2\pi i n t}} \, dt = n. \quad //$$

Example $U = \mathbb{C}$; since γ compact $\Rightarrow$ γ bounded, $\exists R \gg 1$ s.t. $\gamma \subset D(0,R)$. We can also assume (since $L(\gamma) < \infty$) that $R > \frac{L(\gamma)}{2\pi}$. Now let $z := 2R$ $(\in \mathbb{R}_{>0})$ and write

$$\left| W(\gamma, 2R) \right| = \left| \frac{1}{2\pi i} \int_\gamma \frac{1}{w - 2R} \, dw \right|$$
$$\leq \frac{L(\gamma)}{2\pi} \left\| \frac{1}{w - 2R} \right\|_\gamma$$
$$< \frac{L(\gamma)}{2\pi} \frac{1}{R} < 1.$$

Since $W(\gamma, 2R) \in \mathbb{Z}$, it must be 0.

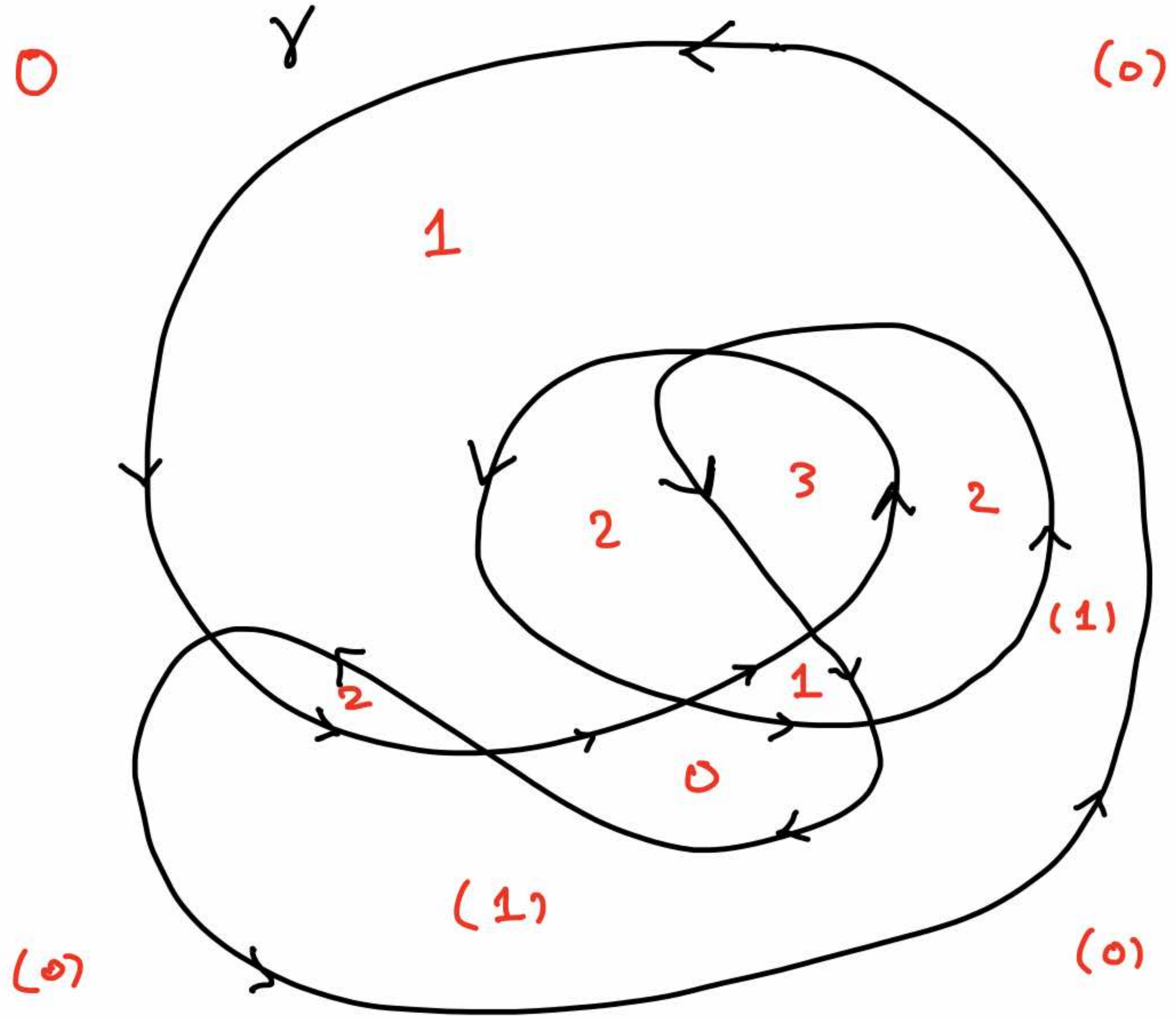
By the Proposition, it is 0 on all of

$(2R\epsilon)$ $U^{ub} :=$ the connected component of $U \setminus \gamma$ containing $D(0,R)^c$.

"unbounded"

Example

Values of $W(\gamma, z)$:



Together with the "strong" version of local Cauchy, the above leads to a slightly different proof of the Cauchy integral formula. (Note that the last Theorem, on which the stuff in this § is based, did not use Cauchy $\int$ formula.) Moreover, <u>we get a statement which permits more general paths:</u>

$\boxed{\text{Theorem (Cauchy integral formula, v. 2.0)}}$

<u>Given $f \in \mathcal{H}ol(D)$, $\gamma \subset D$ closed C^1</u> ← open disk

<u>and $z_0 \in D \setminus \gamma$, we have</u>

actually, result will hold w/o the C^1 assumption

$$\underline{\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz = W(\gamma, z_0)\cdot f(z_0).}$$

Proof: Write $g(z) := \dfrac{f(z)-f(z_0)}{z-z_0}$; since f is continuous, $\lim_{z\to z_0}(z-z_0)\,g(z) \overset{(*)}{=} 0$. By "strong" Cauchy, since $z_0 \notin \gamma$, and $g \in \mathcal{H}ol(D\setminus z_0)$ satisfies $(*)$,

$$0 = \frac{1}{2\pi i}\oint_\gamma g(z)\,dz$$

$$= \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-z_0}\,dz - \underbrace{\frac{1}{2\pi i}\oint_\gamma \frac{f(z_0)}{z-z_0}\,dz}$$

$$= f(z_0)\cdot\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0}$$

$$= f(z_0)\cdot W(\gamma, z_0)$$

which proves the result. $\square$