Lecture 16 : Some interesting functions

I. The complex logarithm

Complementary to the approach in Lecture 9.

Recall that $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$ is star-shaped and therefore simply-connected, and that on a simply-connected region, every holomorphic function has a global holomorphic primitive. Therefore, on $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$, $\frac{1}{z}$ has a global primitive:

$$\log(z) := \int_1^z \frac{1}{w}\,dw.$$

integral is carried out on any path from 1 to z in $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$.

Properties:

- $\log(z) \in \mathcal{H}ol\left(\mathbb{C}\setminus\mathbb{R}_{\leq 0}\right)$
- $\mathrm{Im}(\log(z)) \in (-\pi, \pi)$
- $\log(z)$ is analytic: given $z_1 \in \mathbb{C}\setminus\mathbb{R}_{\leq 0}$, write

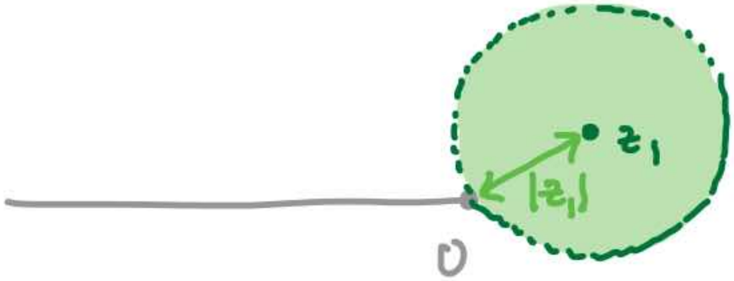
$$\log(z) = \underbrace{\int_1^{z_1} \frac{dw}{w}}_{} + \underbrace{\int_{z_1}^{z} \frac{dw}{w}}_{}$$

[want: const. power series]

↪ will now expand on this:

Let w_1 be such that $e^{w_1} = z_1$, $\arg(w_1) \in (-\pi, \pi)$.

Put $F_{[z_1]}(z) := w_1 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \frac{(z - z_1)^n}{z_1^n}$ on $D(z_1, |z_1|)$.



Now $F'_{[z_1]}(z) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \cdot n \cdot \frac{(z - z_1)^{n-1}}{z_1^n} \underset{m = n-1}{=} \frac{1}{z_1} \sum_{m \geq 0} (-1)^m \left(\frac{z - z_1}{z_1} \right)^m$

$$= \frac{1}{z_1} \cdot \frac{1}{1 + \frac{z - z_1}{z_1}} = \frac{1}{z_1 + z - z_1} = \frac{1}{z} = \log'(z).$$

$\Longrightarrow$ $\log(z) - F_{[z_1]}(z) =: \underset{\text{const.}}{C_{[z_1]}}$

FTC (since F & log both holo.)

$\Longrightarrow$ log analytic at z_1 (and z_1 was arbitrary).
$\left(= C_{[z_1]} + F_{[z_1]}(z)\right)$

Take $z_1 = 1$, $w_1 = 0$ $\Rightarrow$ $\log(1) = 0 = F_{[1]}(1)$

$\Rightarrow$ $C_{[1]} = 0$

$\Rightarrow$ on $D(1,1)$, $e^{\log(z)} = e^{F_{[1]}(z)}$

$= e^{-\sum_{n \geq 1} \frac{(1-z)^n}{n}}$

$\overset{(*)}{=} z$

since both of these are analytic, and they agree on $D(1,1)$, they agree on all of $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

(*) true, as usual, for $z \in (-1,1) \subseteq \mathbb{R}$ (by Calculus) $\Rightarrow$ for $z \in D(1,1)$.

$\Longrightarrow e^{\log z} = z$ on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$

$\Longrightarrow$ on $D(z_1, |z_1|)$, $\exp\left(F_{[z_1]}(z) - \log(z)\right)$

$$= \underbrace{\exp(w_1)}_{\substack{z_1 \\ \text{(by defn.)}}} \cdot \underbrace{\exp\left(-\sum_{n \geq 1} \frac{(1 - z/z_1)^n}{n}\right)}_{z/z_1} \Big/ \underbrace{\exp(\log(z))}_{\substack{\text{just proved} \\ = z}}$$

$= 1$

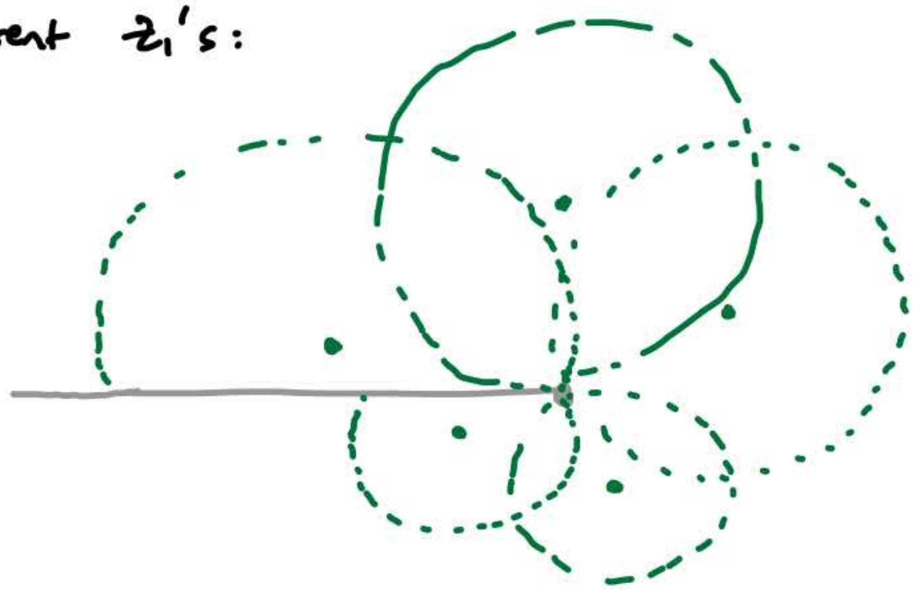
not $2\pi i m$ (since $\arg(w_1) \in (-\pi, \pi)$)

$\Longrightarrow C_{[z_1]} = 0 \quad (\forall z_1)$, i.e.

$\log(z) = F_{[z_1]}(z)$ on $D(z_1, |z_1|)$

(with caveat if $z_1 \in$ left-half plane)

So $\log(z)$ "stitches together" all the power series centered at different z_1's:



See picture

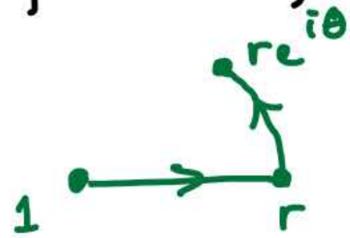
We've just used power-series as an intermediate step, also, between

$$\int_1^z \frac{dw}{w} \longleftrightarrow \underset{\text{"}F\text{"}}{\text{power series}} \longleftrightarrow \text{inverse of } \exp.$$

Formal consequence: for $\theta \in (-\pi, \pi)$,

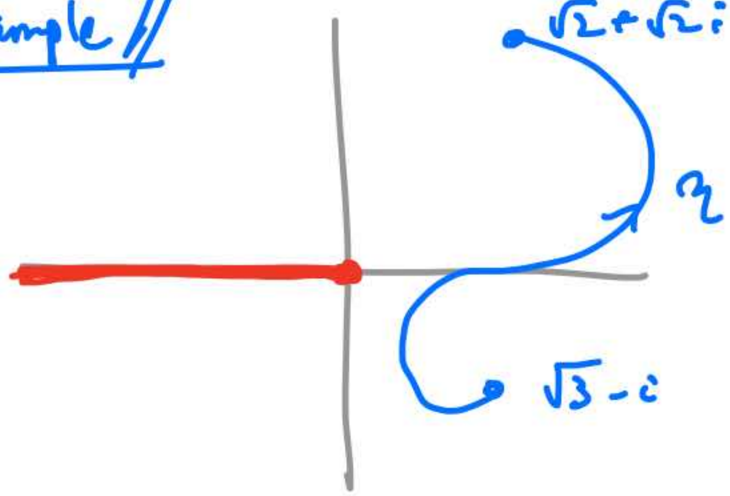
$$\int_1^{re^{i\theta}} \frac{dw}{w} =: \log(re^{i\theta}) = \log(e^{\log r + i\theta}) = \log r + i\theta$$

(or you can just compute this by direct integration along



).

Example



Now without a moment's hesitation you may write

$$\int_\gamma \frac{dw}{w} = \log(w)\Big]_{2e^{-i\pi/6}}^{2e^{i\pi/4}}$$

$$= (\log 2 + i\pi/4) - (\log 2 - i\pi/6)$$

$$= i5\pi/12.$$

More general situations:

① Different branches of $\log(z)$: $\underbrace{U \subset \mathbb{C}\setminus\{0\}}_{z \text{ never } 0 \text{ on } U}$ simply conn.,

$$\log(z) := w_0 + \int_{z_0}^{z} \frac{dw}{w} \quad (\Rightarrow \log(z_0) = w_0)$$

in U

where $e^{w_0} = z_0$

so want

The "branch" depends on U & w (can add $2\pi i m$ here to "change" branch).

② Primitive for $f'(z)/f(z)$ on $U \subset \mathbb{C}$:

U simply-connected, f never 0 on U.

$$\text{“}\log(f(z))\text{”} := w_0 + \int_{z_0}^{z} \frac{df}{f}, \quad \text{where } e^{w_0} = f(z_0).$$

in U

i.e. $\frac{f'(w)}{f(w)}\,dw$

WARNING: this is NOT $f(z)$ plugged into $\log(\cdot)$,

or $\int_{f(z_0)}^{f(z)} \frac{dw}{w}$. Reason: can't necessarily define

log on $f(U)$, because

U simply connected $\not\Rightarrow$ $f(U)$ simply connected!

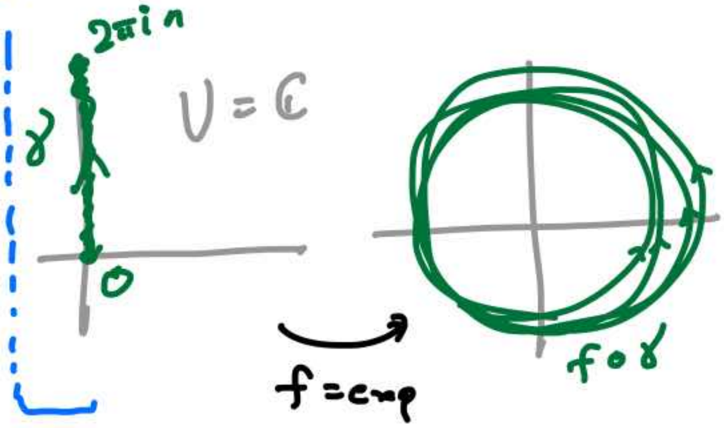
Example// $f(z) = \exp(z)$, $U = \mathbb{C}$. The thing called

“$\log(f(z))$” above is a primitive on all of $\mathbb{C}$ for

$f'(z)/f(z) = e^z/e^z = 1$, i.e. z.

$\Rightarrow \log(f(z)) = z$, $\underbrace{\log(f(2\pi i n)) = 2\pi i n}$.

deceptive notation, as there's no single branch of log itself that takes all these values!!



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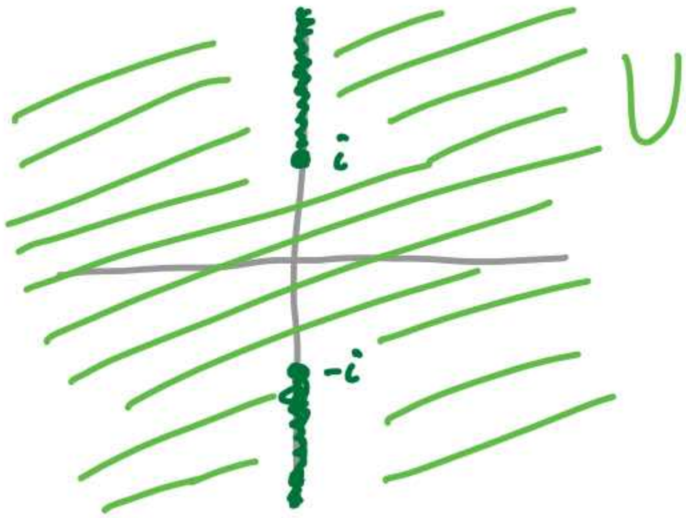
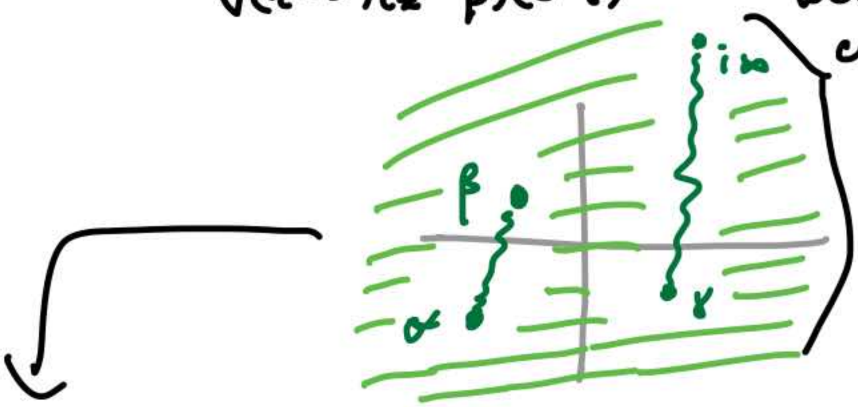
③ Other functions :

(a) $\frac{1}{1+z^2}$ is holomorphic on $\mathbb{C}\setminus\{\pm i\}$ but this isn't simply connected. To make it so, remove 2 rays :

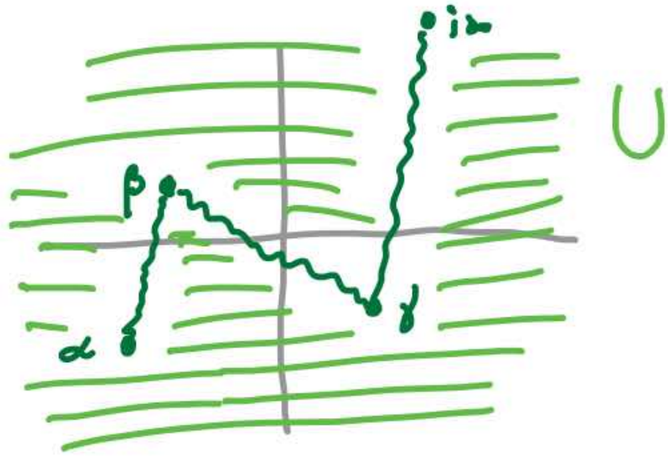


Then $\int_0^z \frac{dw}{1+w^2} =: \arctan(z) \in \mathcal{H}ol(U)$.

(b) $f(z) = \frac{1}{\sqrt{(z-\alpha)(z-\beta)(z-\gamma)}}$ is holomorphic & well-defined on the complement of cuts



But this region isn't simply connected. To fix this, remove a strip from β to γ :

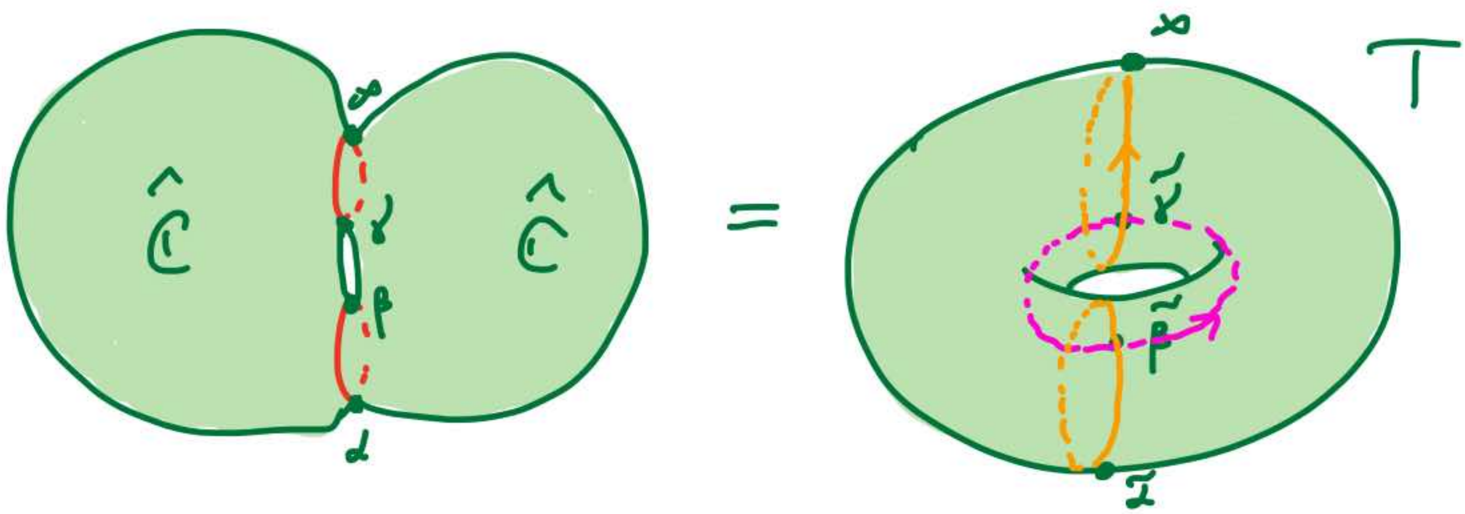


Then the abelian integral

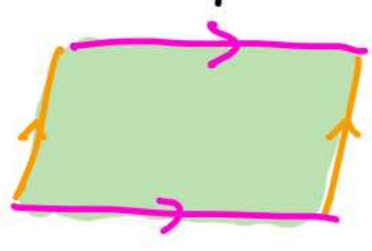
$$A(z) := \int_{z_0}^{z} \frac{dw}{\sqrt{(w-\alpha)(w-\beta)(w-\gamma)}}$$

is well-defined & holomorphic on U.

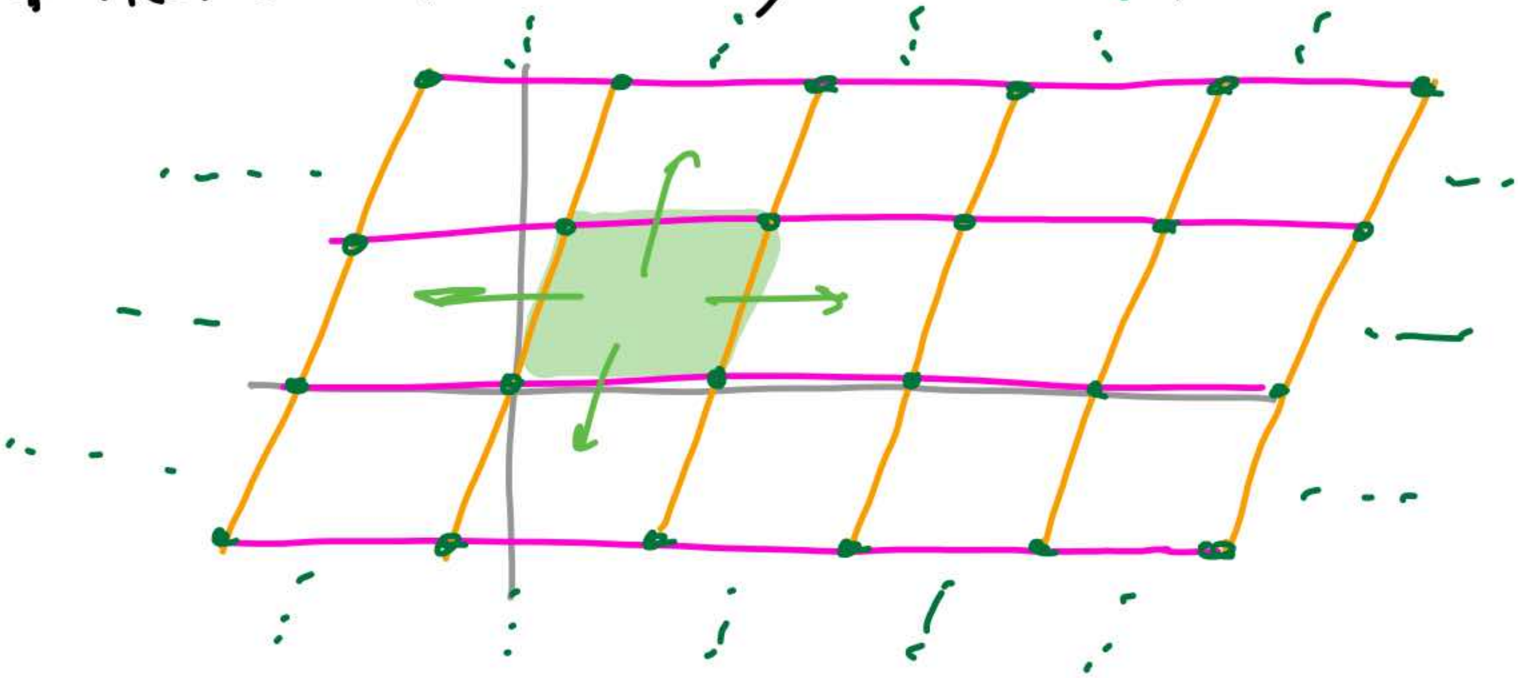
On the Riemann surface of $f(z)$,

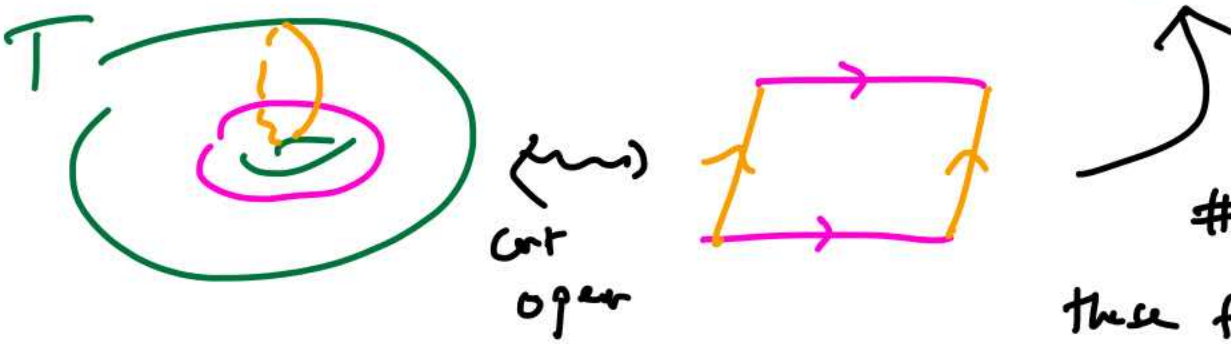


this corresponds to cutting open the torus as shown, but one can omit the cuts from $\tilde{\alpha}$ to $\tilde{\beta}$ without sacrificing simple connectivity. This produces a "parallelogram"



corresponding to the fact that the "existence domain of $A(z)$ over T" is actually the complex plane:





take ∞ # of copies: these fit together to tile $\mathbb{C}$, and are required because A is multivalued "in both horizontal & vertical directions".

The dots form a lattice $\Lambda \cong \mathbb{Z}\langle \lambda_1, \lambda_2 \rangle \subset \mathbb{C}$ and there is a complex analytic isomorphism from $\mathbb{C}/\Lambda$ to T. We'll see this from a different angle (and more clearly) later on in the course.

II. The dilogarithm

$\log(z) \in \mathcal{H}ol(\mathbb{C} \setminus \mathbb{R}_{\leq 0}) \Rightarrow \log(1-z) \in \mathcal{H}ol(\mathbb{C} \setminus \mathbb{R}_{\geq 1})$

$\Rightarrow -\dfrac{\log(1-z)}{z} \in \mathcal{H}ol(\mathbb{C} \setminus \mathbb{R}_{\geq 1})$

Since we divided by z, check analyticity ($\Leftrightarrow$ holomorphicity) at 0 :

$-\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$

$\Rightarrow -\frac{\log(1-z)}{z} = 1 + \frac{z}{2} + \frac{z^2}{3} + \cdots$

Since $\mathbb{C} \setminus \mathbb{R}_{\geq 1}$ is simply connected, $-\dfrac{\log(1-z)}{z}$ has a primitive there:

$$\mathrm{Li}_2(z) := -\int_0^z \log(1-w) \underbrace{d\log(w)}_{:= \frac{dw}{w}}$$

integrate over path in $\mathbb{C} \setminus \mathbb{R}_{\geq 1}$.

Put $\rho(z) := \mathrm{Li}_2(z) - \mathrm{Li}_2(1-z) \in \mathcal{H}ol(\underbrace{\mathbb{C} \setminus [1,\infty) \cup (-\infty, -1]}_{=: U})$

(or $R(z) := \mathrm{Li}_2(z) + \frac{1}{2}\log(z)\log(1-z)$)

very similar function

Then

$$d(\rho(z)) = -d\int_0^z \frac{\log(1-w)}{w}\,dw + d\int_0^{1-z} \frac{\log(1-w)}{w}\,dw$$

$$= -\frac{\log(1-z)}{z}\,dz - \frac{\log(z)}{1-z}\,dz$$

$$= -\log(1-z)\,d\log z + \log(z)\,d\log(1-z).$$

Consider now 5 points $z_1, \dots, z_5 \in \mathbb{C}$ such that the "cyclically permuted cross ratios"

$$\left\{\begin{aligned} \alpha_1 &= CR(z_1, z_2, z_3, z_4)^{-1} \\ \alpha_2 &= CR(z_2, z_3, z_4, z_5)^{-1} \\ \alpha_3 &= CR(z_3, z_4, z_5, z_1)^{-1} \\ \alpha_4 &= CR(z_4, z_5, z_1, z_2)^{-1} \\ \alpha_5 &= CR(z_5, z_1, z_2, z_3)^{-1} \end{aligned}\right\} \text{ belong to } U.$$

(This is true for a "generic" choice of z_i's.)

$\boxed{\textbf{Theorem (the "5-term relation")}}$ $\underline{\displaystyle\sum_{i=1}^{5} \rho(\alpha_i) = \frac{\pi^2}{6},}$

<u>independently of the $\{z_i\}$.</u> (Alternately: $\sum R(\alpha_i) = \frac{\pi^2}{2}$.)

Proof: (Recall $CR(a,b,c,d) := \frac{a-c}{a-d}\cdot\frac{b-d}{b-c}$, so $CR(a,b,c,d)^{-1} = \frac{(a-d)(b-c)}{(a-c)(b-d)}$.)

Step 1: With indices mod 5,

$$(*) \quad \boxed{1 - \alpha_i = \alpha_{i-1}\alpha_{i+1} \quad (\forall i)} \ :$$

$$1-\alpha_i = 1 - \frac{z_i - z_{i-2}}{z_i - z_{i+2}}\cdot\frac{z_{i+1}-z_{i+2}}{z_{i+1}-z_{i-2}}$$

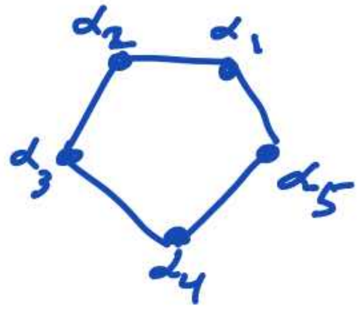
$$= \frac{\left\{\begin{array}{l}\cancel{z_i z_{i+1}} - z_i z_{i-2} - z_{i+2}z_{i+1} + \cancel{z_{i+2}z_{i-2}} \\ \quad -\cancel{z_i z_{i+1}} + z_i z_{i+2} + z_{i-2}z_{i+1} - \cancel{z_{i-2}z_{i+2}}\end{array}\right\}}{(z_i - z_{i+2})(z_{i+1}-z_{i-2})}$$

$$= \frac{(z_i - z_{i+1})(z_{i+2}-z_{i-2})}{(z_i - z_{i+2})(z_{i+1}-z_{i-2})}$$

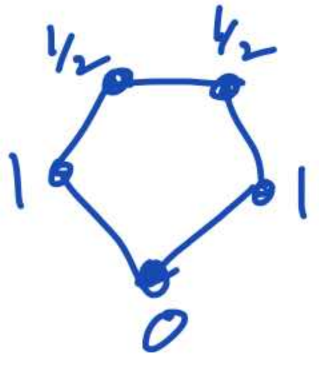
$$= \left(\frac{\cancel{z_{i-1}-z_{i+2}}}{\cancel{z_{i-1}-z_{i+1}}}\cdot\frac{z_i - z_{i+1}}{z_i - z_{i+2}}\right)\left(\frac{\cancel{z_{i+1}-z_{i-1}}}{z_{i+1}-z_{i-2}}\cdot\frac{z_{i+2}-z_{i-2}}{\cancel{z_{i+2}-z_{i-1}}}\right)$$

$$= \alpha_{i-1}\cdot\alpha_{i+1}.$$

Step 2: I like to use the notation



to mean $(*)$. So one has



$\rightsquigarrow$ $\rho(1/2) = 0$, $\rho(1) = \pi^2/6$, $\rho(0) = -\pi^2/6$ $\rightsquigarrow$ $2\rho(\frac{1}{2}) + 2\rho(1) + \rho(0) = \frac{\pi^2}{6}$, so works for this configuration!

Step 3: Left to check that as a function of the $\{z_i\}$, $\underbrace{\sum \rho(\alpha_i) = \text{constant}}$:

$$d \sum_{i=1}^{5} \rho(\alpha_i) = \sum_{i=1}^{5} d\rho(\alpha_i)$$

$$= \sum_{i=1}^{5} \log(\alpha_i)\, d\log \underbrace{(1-\alpha_i)}_{\alpha_{i-1}\alpha_{i+1}} - \sum_{i=1}^{5} \log \underbrace{(1-\alpha_i)}_{\alpha_{i-1}\alpha_{i+1}} d\log(\alpha_i)$$

$$= \left\{ \sum \log(\alpha_i)\, d\log(\alpha_{i-1}) - \sum \log(\alpha_{i+1})\, d\log(\alpha_i) \right\}$$

$$+ \left\{ \sum \log(\alpha_i)\, d\log(\alpha_{i+1}) - \sum \log(\alpha_{i-1})\, d\log(\alpha_i) \right\}$$

each bracketed term is zero by Δ of index on the right-hand sums

$$= 0 + 0 = 0.$$

$\square$