Lecture 15: Cauchy's Theorem (I)

I. Local Cauchy Theorem

In the last lecture, we proved that for a disk D

$$(*) \quad \boxed{\left.\begin{array}{l} f \in \mathcal{H}ol(D) \\ \gamma \subset D \text{ loop} \end{array}\right\} \Longrightarrow \int_\gamma f\, dz = 0.}$$

What follows is a refinement of this result.

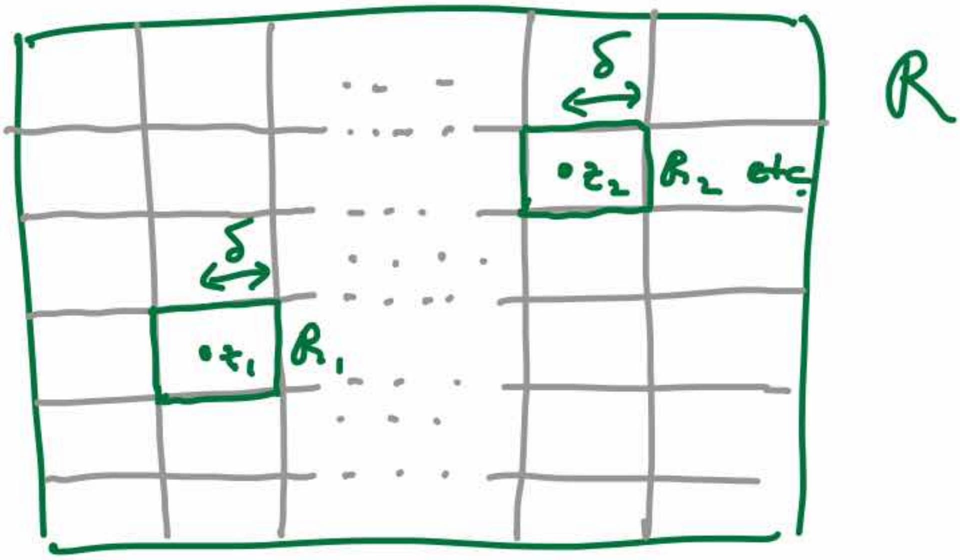
Lemma Let
- $R \subset \mathbb{C}$ be a closed rectangle
- $f \in \mathcal{H}ol(R \setminus \{z_1, \dots, z_n\})$ s.t.
- $\lim_{z \to z_j} (z - z_j) f(z) = 0 \quad (\forall j)$.

Then $\int_{\partial R} f(z)\, dz = 0$.

Proof: Let $\epsilon > 0$ be given, and take δ sufficiently small that

$$|z - z_j| < \sqrt{2}\,\delta \Longrightarrow |(z - z_j) f(z)| < \epsilon/n.$$

Let R_j be a square centered at z_j of side length 2δ, and decompose R into sub-rectangles as shown:



Then $$\left| \int_{\partial R} f\,dz \right| \underset{\text{Goursat}}{=} \left| \sum_{j=1}^{n} \int_{\partial R_j} f\,dz \right|$$

$$\leq \sum_{j=1}^{n} L(\partial R_j) \|f\|_{\partial R_j}$$

$$= \sum_{j=1}^{n} 8\delta \left\| \frac{(z-z_j)f(z)}{(z-z_j)} \right\|_{\partial R_j}$$

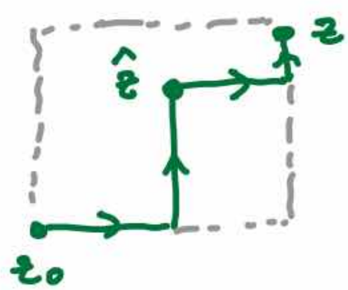
$$\leq \frac{8\delta\epsilon}{n} \sum_{j=1}^{n} \underbrace{\left\| \frac{1}{z-z_j} \right\|_{\partial R_j}}_{1/\delta}$$

$$= 8\epsilon. \qquad \square$$

Writing $D = D(z_0, r)$ and $U = D \setminus \{z_1, \dots, z_n\}$, we then have the

$\boxed{\text{Theorem}}$ $f \in \mathcal{H}ol(U)$, $\lim\limits_{z \to z_j} (z - z_j) f(z) = 0$ $(\forall j)$, $\gamma \subset U$ closed C^1 path $\Longrightarrow \int_\gamma f(z)\,dz = 0$.

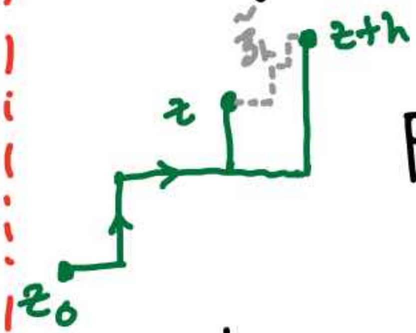
Proof: As in Lecture 14, construct a primitive for f on U: given $z \in U$, let

$\xi_z :=$ 

for *any* choice of $\hat{z}$ within the (closed) dotted box, subject only to the condition that the path avoid all the $\{z_j\}_{j=1}^n$. The difference between any two choices consists of a sum of boundaries of boxes (chosen to avoid $\{z_j\}_{j=1}^n$). By the Lemma,

$$F(z) := \int_{\xi_z} f(z)\,dz$$

is well-defined. Once again, one shows it is holomorphic (with derivative f) by computing



$$F(z+h) - F(z) = \sum_i \int_{\partial R_i} f(w)\,dw + \int_{\tilde{\xi}_h} f(w)\,dw$$

and the proof proceeds as in Lecture 14. $\square$

II. Homotopy Cauchy theorem

Let $U \subset \mathbb{C}$ be open, $\gamma \subset U$ a path (i.e. $\gamma : [a,b] \to U$ of class C^0). For each $z \in U$, $\exists$ $\epsilon > 0$ s.t. $D(z,\epsilon) \subset U$.

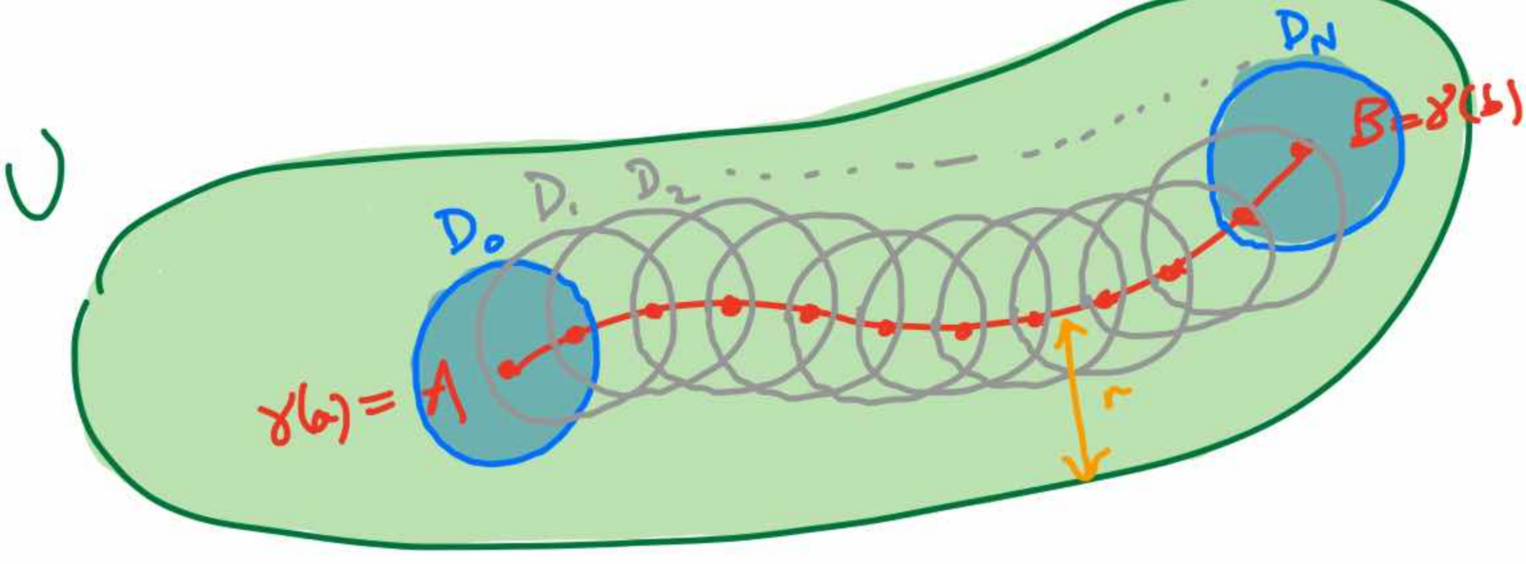
Hence $d(z, U^c) := \inf_{w \in U^c} |z-w| > 0$ defines a (continuous!) function $d : U \to \mathbb{R}_+$. <u>Since $[a,b]$ is compact</u>, $d \circ \gamma$ (also continuous!) achieves its glb $=: r$. So r is a value taken by $d \circ \gamma$ hence <u>lies in $\mathbb{R}_+$</u>, with the consequence that

$$(**) \quad \boxed{D(\gamma(t), r) \subset U \text{ for every } t \in [a,b].}$$

Compactness of $[a,b]$ also tells us that γ is uniformly continuous, i.e. given $\epsilon \in (0, r/2)$ $\exists\, \delta > 0$ s.t.

$$|s-t| < \delta \implies |\gamma(s) - \gamma(t)| < \epsilon.$$
$$(s, t \in [a,b])$$

Let $\Pi = \{\underset{a}{\overset{}{\underset{\shortparallel}{t_0}}} < \dots < \underset{b}{\underset{\shortparallel}{t_N}}\}$ be a partition of $[a,b]$ with $|t_i - t_{i-1}| < \delta$ $(\forall i)$. Writing $I_i = [t_i, t_{i+1}]$, we have $\gamma(I_i) \subset \underbrace{D(\gamma(t_i), \epsilon)}_{D_i} \subset U$.



Now keep in mind that γ is C^0 — not even assumed rectifiable.

Definition Given $f \in \mathcal{H}ol(U)$, let

- Π be a partition of $[a,b]$ into intervals I_i
- $D_i \subset U$ be disks with $\gamma(I_i) \subset D_i$
- $F_i \in \mathcal{H}ol(D_i)$ be primitives of $f|_{D_i}$

in this order

we just demonstrated existence

exist by $(*)$

and set

$$(\#) \quad \boxed{\int_\gamma f(z)dz := \sum_{i=0}^{N-1} \left(F_i(z_{i+1}) - F_i(z_i)\right).}$$

$\boxed{Q1}$ If γ is rectifiable, does this give the right answer?

YES — this is essentially the fundamental theorem of calculus. Suffices to check in one disk:

$$\lim_{|\Pi|\to 0} \left| \sum_i f(\gamma(t_i))(\Delta\gamma)_i - (F(B)-F(A)) \right| \leq \lim_{|\Pi|\to 0} \sum_i \left| f(\gamma(t_i))(\Delta\gamma)_i - (F(t_{i+1})-F(t_i)) \right|$$

$$= \lim_{|\Pi|\to 0} \sum |\Delta\gamma|_i \, |\varepsilon_i| = 0.$$

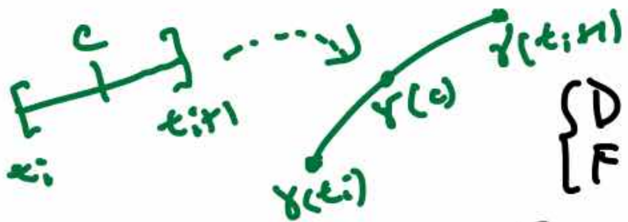
$F(t_{i+1}) - F(t_i) = (\Delta\gamma)_i \left(f(t_i) + \varepsilon_i\right)$, $\varepsilon_i \to 0$ with $|\Pi|$ (uniformly in i).

Q2 If γ isn't rectifiable†, is this well-defined?

YES — independence of choice of the $\{F_i\}$ (on D_i) is trivial

— independence of choice of $\{D_i\}$ likewise (if the $\{I_i\}$ are fixed)

— issue is what happens when there are two different partitions Π, Π'. Since any 2 have a common refinement, we must check that (4) is invariant under refinement. We can do this on one interval and choose any D, F we like:



here we just take $\begin{cases} D \\ F \end{cases}$ to be $\begin{cases} D_i \\ F_i \end{cases}$ for both $[t_i, c]$ and $[c, t_{i+1}]$, so (4) doesn't change.

Definition Paths $\eta, \gamma : [a,b] \to U$ are <u>close together</u> $\iff$ $\exists \begin{cases} \Pi \subset [a,b] \\ D_i \subset U \end{cases}$ s.t.
(open disks)

$\gamma(I_i), \eta(I_i) \subset D_i \; (\forall i)$.

---

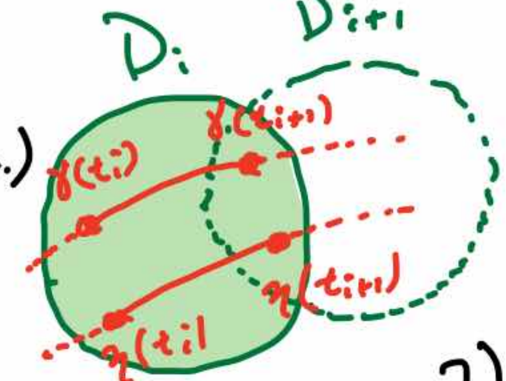
† e.g. $\gamma(t) = t + i t \sin\left(\frac{1}{t}\right)$
$t \in [0,1]$.

In the free abelian group on points of $\mathbb{C}$, write

$$\partial\gamma := [\gamma(b)] - [\gamma(a)].$$

(This is $0 \iff \gamma$ is closed.)

$\boxed{\text{Lemma}}$ Let $\gamma, \eta \subset U$ be close together, with $\partial\gamma = \partial\eta$, and $f \in \mathcal{H}ol(U)$. Then $\int_\gamma f\,dz = \int_\eta f\,dz$.



Proof: CASE 1: $\partial\gamma, \partial\eta \neq 0$
($\Rightarrow$ have same start/endpt.)

$$\int_\gamma f\,dz - \int_\eta f\,dz =$$

$$\sum_{i=0}^{n-1} \left\{ F_i(\gamma(t_{i+1})) - F_i(\gamma(t_i)) - \left[F_i(\eta(t_{i+1})) - F_i(\eta(t_i)) \right] \right\} =$$

$$\sum_{i=0}^{n-1} \left\{ \left[F_i(\gamma(t_{i+1})) - F_i(\eta(t_{i+1})) \right] - \left[F_i(\gamma(t_i)) - F_i(\eta(t_i)) \right] \right\} =$$

$i \to i+1$ (change both)

collapse

$$F_n(\gamma(b)) - F_n(\eta(b)) - F_0(\gamma(a)) + F_0(\eta(a)) \quad (*)$$

$=$ since $\partial\gamma = \partial\eta$ $=$

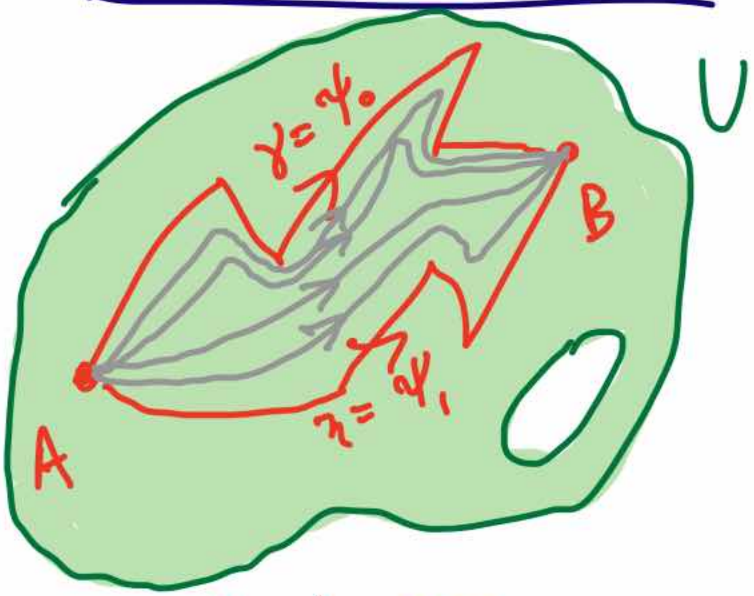
$$= 0.$$

CASE 2: $\partial\gamma = 0 = \partial\eta$. Then $\gamma(b)$ may NOT equal $\eta(b)$ [resp. $\gamma(a)$ . . . . $\eta(a)$]

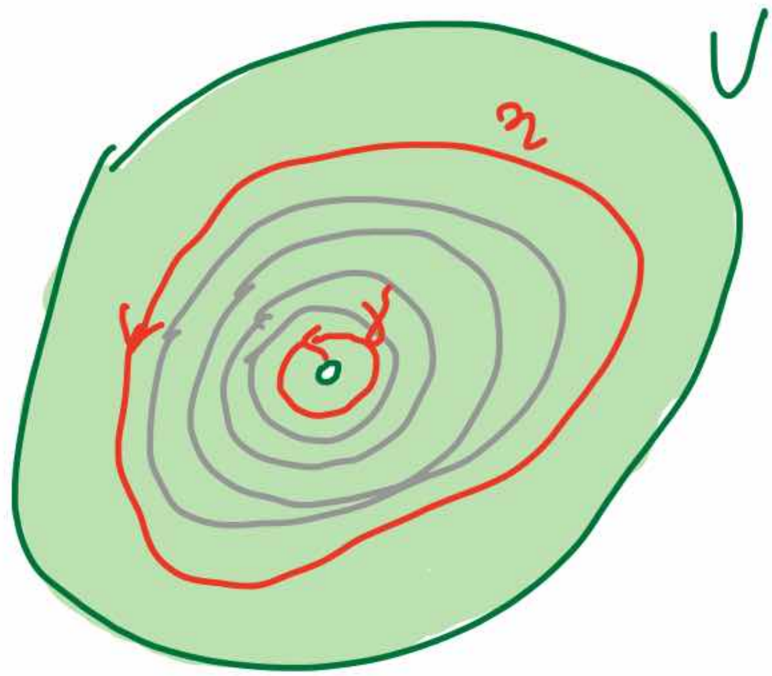
but $\begin{cases}\gamma(b) = \gamma(a) \\ \eta(b) = \eta(a)\end{cases}$, and can take $F_n = F_0$, $D_n = D_0$. Again $(*) = 0$. $\square$

As above, let $\gamma, \eta : [a,b] \to U$ be (C^0) paths with $\underset{\sim}{\partial\gamma = \partial\eta}$.

Definition γ is homotopic to η in U "$\gamma \sim \eta$" $\Longleftrightarrow$
$\exists\ C^0\ \psi : [a,b]\times[0,1] \longrightarrow U$ with
$\psi(t,0) = \gamma(t)$, $\psi(t,1) = \eta(t)$, and (writing $\psi(t,s) =: \psi_s(t)$)
$\partial\psi_s = \partial\gamma\ (= \partial\eta)\ \forall s \in [0,1]$.



non-closed case



closed case

Theorem γ, η homotopic, $f \in \mathcal{H}ol(U)$

$$\Longrightarrow \int_\gamma f\,dz = \int_\eta f\,dz.$$

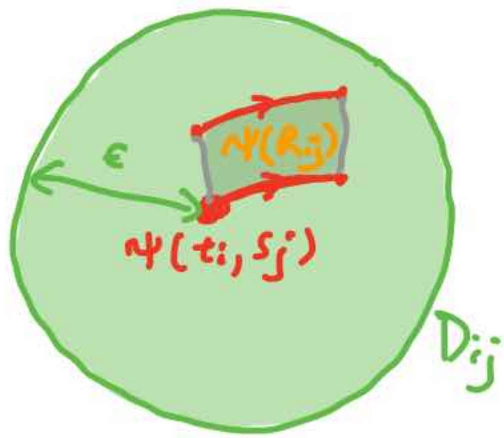
Proof: Let Π, P be partitions of $[a,b]$ resp. $[0,1]$ s.t.

I_i's, J_j's $\rightsquigarrow R_{ij} := I_i \times J_j$

$\psi(R_{ij}) \subset D(\psi(t_i, s_j), \epsilon) =: D_{ij}$ where $\epsilon \in (0, d(\psi(R_{ij}), U^c))$.

These exist by uniform continuity.

Clearly, ψ_{s_j} and $\psi_{s_{j+1}}$ are close together for each j, and have the same "boundary" $\partial(\cdot)$. Hence the Lemma $\Rightarrow \int_{\psi_0} f\,dz = \int_{\psi_1} f\,dz = \cdots = \int_{\psi_n} f\,dz$. $\square$



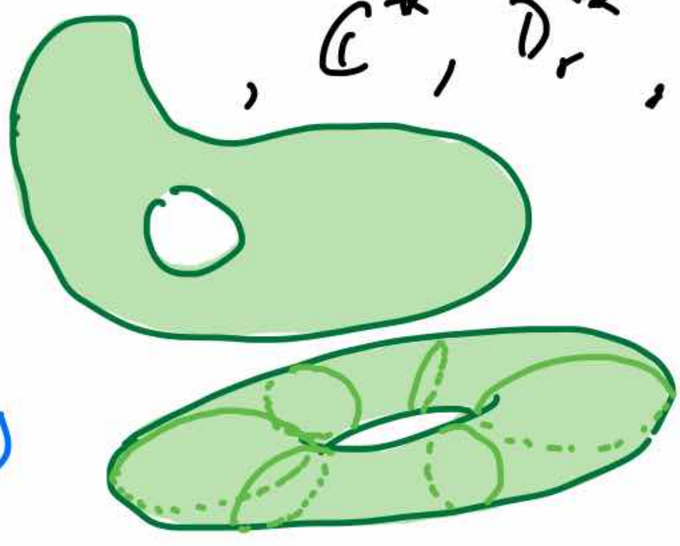
III. Simple-connectedness

Let $U \subset \mathbb{C}$ be open.

U is simply connected $\overset{\text{def.}}{\iff}$ every (C^0) closed path $\gamma \subset U$ is homotopic (in U) to a point, i.e. constant path.

Non-Examples

(subsets of $\mathbb{C}$)



, $\mathbb{C}^*$, D_r^*, $\mathbb{C} \setminus [-1, 1]$, etc.

(Riemann surfaces)

Examples

① $\mathbb{C}$ itself (or $\hat{\mathbb{C}}$ if you want something compact)

② convex subsets: if $z, w \in U$, then $sz + (1-s)w \in U$
($\forall s \in [0,1]$)

③ star-shaped sets (HW): $\exists z_0 \in U$ s.t. given any $z \in U$, the segment $\overline{z_0 . z} \subset U$

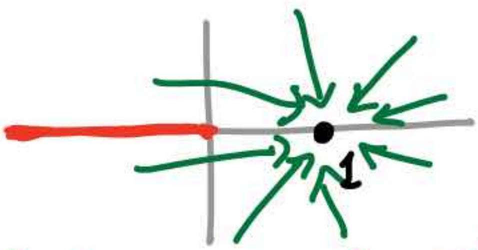
④ disk is convex & star-shaped; $\mathbb{C} \setminus \mathbb{R}_{\le 0}$ is star-shaped ($z_0 = 1$), as is $\mathbb{C} \setminus (-\infty, -1] \cup [1, \infty)$.

Proof that (i) $\mathbb{C}$, (ii) convex sets, (iii) $\mathbb{C} \setminus \mathbb{R}_{\le 0}$ are simply-connected.

(i) Given (C^0) $\gamma : [a,b] \to \mathbb{C}$ closed, define $\Psi : [a,b] \times [0,1] \to \mathbb{C}$ by $\Psi(t,s) := (1-s) \cdot \gamma(t)$ ("contraction to 0")

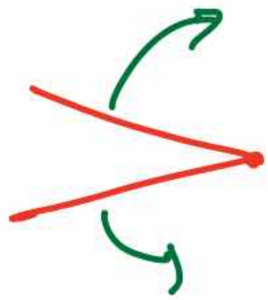
(ii) Let U be convex. Can contract γ to its own start/end pt.:
$\Psi(t,s) := s \cdot \gamma(a) + (1-s) \cdot \gamma(t)$.

(iii)



$\left(\begin{array}{c} \Psi(t,s) = s + (1-s)\gamma(t) \\ \text{"contract to 1"} \end{array} \right)$

OR



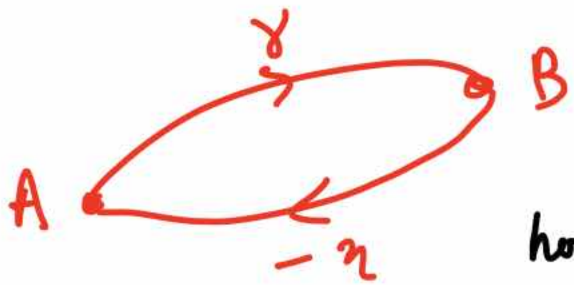
"fold back" and contract distance to 1 or $r(a)$

writing $\gamma(t) = r(t) e^{i\theta(t)}$

$\Psi(t,s) := \left(s \cdot r(a) + (1-s) r(t) \right) e^{i\theta(t)(1-s)}$

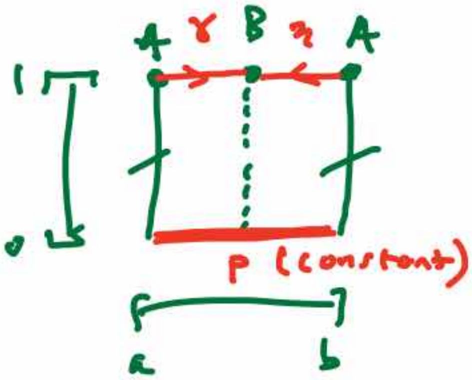
[key point is: $\theta(a) = \theta(b)$, not just mod $2\pi\mathbb{Z}$] □

<u>Remark</u>: Homotopy of paths based at A & B is clearly transitive. Furthermore, if U is simply connected, then for $\gamma \sim \eta$ (with $\partial\gamma = [B]-[A] = \partial\eta$),



is closed and homotopic to a point p.

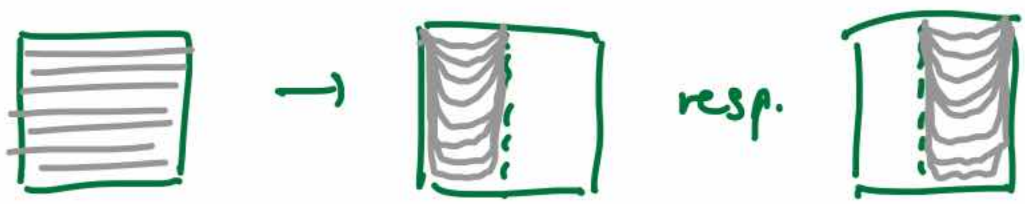
Say Ψ as here gives the homotopy:



Then Ψ takes the same value on the marked vertical segments; hence Ψ of these two paths

have precisely the same image, call it Ω.

Composing Ψ with a function from $[0,1]^2$ to $[a,b)\times[0,1]$ sending



gives $\gamma \sim \Omega \sim \eta$.

So in simply-connected U, $\partial\gamma = \partial\eta \Rightarrow \gamma \sim \eta$. //

Why are we doing this? To get the following corollary to the "homotopy Cauchy theorem" above:

Corollary 1 U simply connected, $\gamma \subset U$ closed, $f \in \mathcal{Hol}(U) \Rightarrow \int_\gamma f\, dz = 0$.

Proof: $\gamma \sim$ pt. (i.e. constant path η), and

$\int_\eta f\, dz = 0$ since $\eta'(t) = 0$. $\square$

Corollary 2 U simply connected $\Rightarrow$ any $f \in \mathcal{Hol}(U)$ has a global holomorphic primitive on U, namely $F(z) := \int_{z_0}^{z} f(w)\, dw$ (+ const.)

Proof: F is well-defined because any 2 paths in U from z_0 to z are homotopic in U (so apply homotopy Cauchy). F differentiable with $F' = f$ is as usual. $\square$