Lecture 14: Prelude to Cauchy

I. Cauchy's actual theorem

We are aiming at proving from scratch various versions of Cauchy's Theorem, which basically says that if $f \in \mathcal{H}ol(U)$, U a simply connected region, then $\int_\gamma f\,dz = 0$ for γ closed. (We want simple connectivity so that the region enclosed by γ is in U.) <u>If</u> we assume f is not just $\mathbb{C}$-differentiable but <u>continuously</u> so†, this can be done using Green's Theorem from vector calculus (hence not quite from scratch): let $\vec{F}(x,y)$ be a vector field of class C^1 on U. Then

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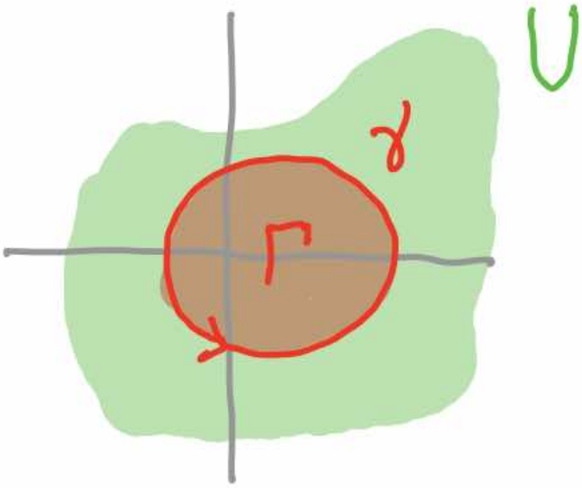
†...by which we mean that u_x, u_y, v_x, v_y are C^0. This is (as an assumption) undesirable: while it will turn out to be the case that $f \in \mathcal{H}ol(U) \Rightarrow f \in \mathcal{A}n(U)$ (a fortiori continuously differentiable), it will be <u>Cauchy</u> that <u>enables us to show this</u>!!



$$\int_\gamma \vec{F}\cdot d\vec{\ell} \overset{(*)}{=} \iint_\Gamma \operatorname{curl}(\vec{F})\,dx\,dy.$$

oriented counter-clockwise

$\partial\Gamma = \gamma$

$$d\vec{\ell} = \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix} dt$$

where $\vec{\gamma}(t) = \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix}$

$\operatorname{curl}(\vec{F}) = \partial F_y/\partial x - \partial F_x/\partial y$

Now
$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+i\,dy)$$

i.e. $\gamma'(t)\,dt = \gamma_1'(t)\,dt + i\,\gamma_2'(t)\,dt$

$$= \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy)$$

$$= \int_\gamma \begin{pmatrix} u \\ -v \end{pmatrix}\cdot d\vec{\ell} + i\int_\gamma \begin{pmatrix} v \\ u \end{pmatrix}\cdot d\vec{\ell}$$

$$\overset{\text{by }(*)}{=} \iint_\Gamma \underbrace{-\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)}\,dx\,dy + i\iint_\Gamma \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}\,dx\,dy$$

both 0 by C-R eqns

$$= 0.$$

This is essentially Cauchy's original result, but due to more recent improvements in technology, it will not be our "official" Cauchy Theorem.

II. Goursat's Theorem

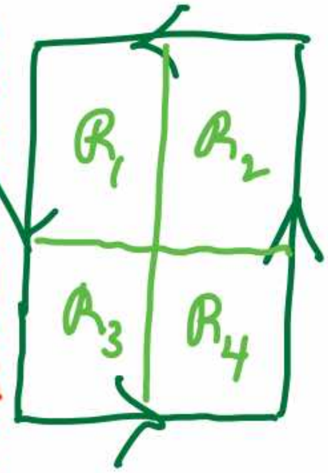
Here is one of those technical improvements: we only assume that the complex derivative exists at each point. (This is significant, because we eventually want to prove that continuous differentiability follows from differentiability.)

Theorem (Goursat, 1884) Given a closed rectangle $R \subset \mathbb{C}$ and $f \in \mathrm{Hol}(R)$,

$$\int_{\partial R} f \, dz = 0.$$

recall that this means $\exists$ open $U \supset R$ s.t. f extends to a holo. fcn. on U

Proof: $R = \bigcup_{i=1}^{4} R_i \implies$



$$\int_{\partial R} f \, dz = \sum_{i=1}^{4} \int_{\partial R_i} f \, dz \implies$$

$$\left| \int_{\partial R} f \, dz \right| \leq \sum_{i=1}^{4} \left| \int_{\partial R_i} f \, dz \right|$$

$$\leq 4 \left| \int_{\partial R^{(1)}} f \, dz \right|$$

pick one of the four

$$\leq 4^2 \left| \int_{\partial R^{(2)}} f \, dz \right|$$

subdivide $R^{(1)}$ & repeat

$$\leq \cdots\cdots$$

$$\leq 4^n \left| \int_{\partial R^{(n)}} f \, dz \right| .$$

The sequence $z_i := \text{center}(R^{(i)})$ is Cauchy, hence has a limit point which lies in every $R^{(n)}$ (since they are compact and contain the tails). The intersection cannot contain more than one point and so

$$\bigcap_{i=1}^{\infty} R^{(i)} = \lim_{i \to \infty} z_i =: z_\infty .$$

Now
$$\int_{\partial R^{(n)}} f(z)\, dz = \underbrace{\cancel{\int_{\partial R^{(n)}} f(z_\infty)\, dz}}_{\to 0} + f'(z_\infty) \underbrace{\cancel{\int_{\partial R^{(n)}} (z - z_\infty)\, dz}}_{\to 0}$$
$$+ \int_{\partial R^{(n)}} (z - z_\infty) H(z)\, dz ,$$

(since $1, z$ have holo. primitives on all of $\mathbb{C}$)

where
$$H(z) := \begin{cases} \dfrac{f(z) - f(z_\infty)}{z - z_\infty} - f'(z_\infty) , & z \neq z_\infty \\ 0 , & z = z_\infty \end{cases}$$

is continuous by the definition of complex differentiability at z_∞ (recall f is assumed holomorphic). So

$$\left| \int_{\partial R} f\, dz \right| \le 4^n \left| \int_{\partial R^{(n)}} (z - z_\infty) H(z)\, dz \right|$$

$$\le 4^n L(\partial R^{(n)}) \| z - z_\infty \|_{\partial R^{(n)}} \| H(z) \|_{\partial R^{(n)}}$$

$$= \cancel{4^n} \cdot \frac{1}{\cancel{2^n}} L(\partial R) \cdot \frac{1}{\cancel{2^n}} \operatorname{diam}(R) \cdot \| H(z) \|_{\partial R^{(n)}}$$

$$= \text{const.} \times \| H(z) \|_{\partial R^{(n)}} \xrightarrow[n \to \infty]{} 0$$

since the $R^{(n)}$ are contained in smaller & smaller neighborhoods of z_∞ in U. $\square$

Recall that in the last lecture we had a result which stated that for a <u>continuous</u> function f on a region U, the conditions

$(*)$
- f has a holomorphic primitive F
- $\int_A^B f\, dz$ is independent of path
- $\int_\gamma f\, dz = 0$ for every loop $\gamma \subset U$

were equivalent.

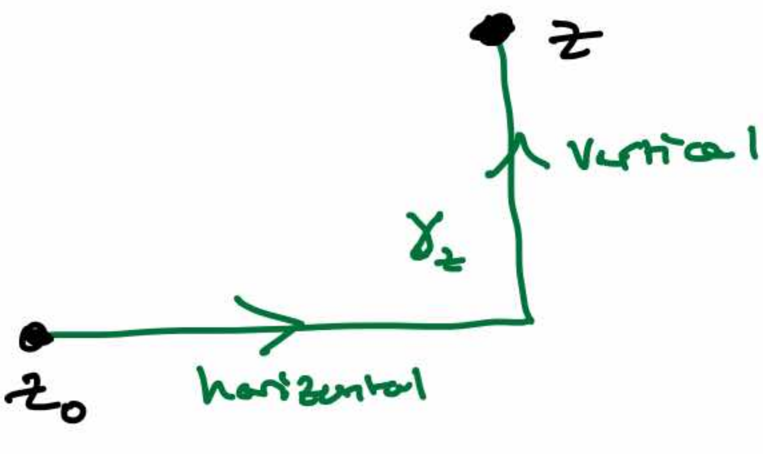
Proposition Let $U = D(z_0, r)$ (= disk).

(a) If $f \in C^0(U)$ has $\int_{\partial R} f\,dz = 0$ ($\forall$ rectangles $R \subset U$) then the conditions $(*)$ hold.

(b) If $f \in \mathrm{Hol}(U)$, then the conditions $(*)$ hold.

Proof: (b): Once we know (a), this is an immediate corollary of Goursat.

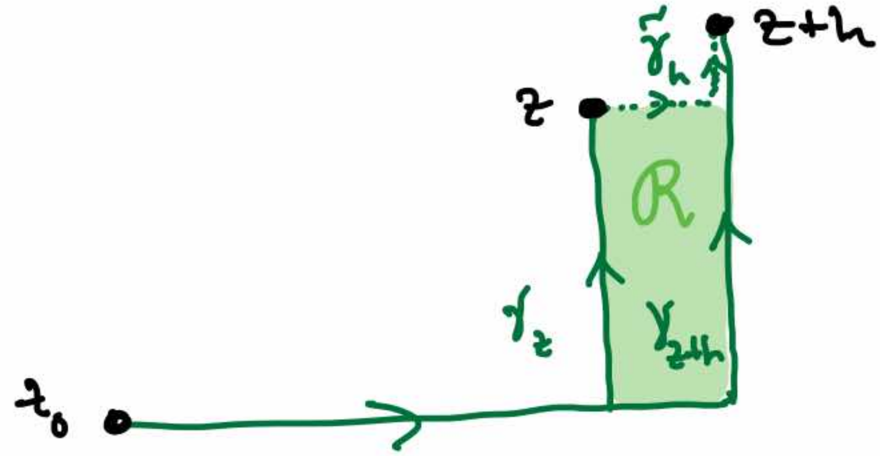
(a): We construct a holomorphic primitive F for f. Given $z \in U$, let γ_z be the path



and set $F(z) := \int_{\gamma_z} f(w)\,dw$. Then

$$\gamma_{z+h} - \gamma_z = \partial R + \tilde{\gamma}_h$$

(see figure)



$$\Longrightarrow \qquad F(z+h) - F(z) = \int_{\tilde{\gamma}_h} f(w)\,dw .$$

$\left(\text{use } \int_{\partial \mathcal{R}} f(w)\,dw = 0\right)$

Trivially $f(z) = \frac{1}{h} \int_{\tilde{\gamma}_h} \overset{\text{constant}}{f(z)}\,dw$, so

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \left| \frac{1}{h} \int_{\tilde{\gamma}_h} \left(f(w) - f(z)\right) dw \right|$$

$$\leq \underbrace{\frac{L(\tilde{\gamma}_h)}{|h|}}_{\leq \sqrt{2}} \, \| f(w) - f(z) \|_{\tilde{\gamma}_h}$$

$\longrightarrow 0$ with h. $\square$

III. Remark on differential forms

Recall that the change-of-variable formula for multiple integrals implies that given $F = u + iv \in C^1(U)$ and $Q \subset U$ compact,

$$\iint_{F(Q)} du\, dv = \iint_Q \underbrace{(u_x v_y - u_y v_x)}_{\det(J_F)} dx\, dy .$$

If $F \in \mathcal{H}ol(U)$ then $u_x v_y - u_y v_x \underset{C-R}{=} u_x^2 + v_x^2$

and then the above $= \iint_Q |F'(z)|^2\, dx\, dy$.

(F holo. $\Rightarrow$ use $\frac{dF}{dz} = \frac{\partial F}{\partial x} = u_x + i v_x$.)

The best way to look at this change-of-variable computation is in terms of <u>differential forms</u>:

for $\mathcal{U} \subset \mathbb{R}^n$,

$$A^k(\mathcal{U}) := \left\{ \begin{array}{l} \text{expressions of the form} \\ \sum_{I} f_I(\underline{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \end{array} \middle| f_I \in C^\infty(\mathcal{U}) \right\}$$

$I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}, \; i_1 < \cdots < i_k$

$$\cong \; C^\infty(\mathcal{U}) \otimes \Lambda^k \mathbb{R}^n.$$

($\wedge$ is "anti commutative": $A \wedge B = -B \wedge A$.)

Defining

$$d\left\{f(\underline{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right\} := \sum_{\ell \in \{1,\ldots,n\} \setminus I} (-1)^{\langle \ell \rangle_I} \frac{\partial f}{\partial x_\ell}\, \overbrace{\underbrace{dx_{i_1} \wedge \cdots \wedge dx_\ell}_{\#\text{ of terms here } =: \langle \ell \rangle_I} \wedge \cdots \wedge dx_{i_k}}^{i_1 < \cdots < \ell < \cdots < i_k}$$

and extending $\mathbb{C}$-linearly gives a map

$$A^k(\mathcal{U}) \xrightarrow[d]{} A^{k+1}(\mathcal{U})$$

called <u>exterior differentiation</u>. The k-dimensional generalization of a C^1 path is a "k-chain" — this is what you want to integrate a k-form over. These come with a <u>boundary map</u>

$$\underset{((k+1)\text{-chains})}{C_{k+1}(\mathcal{U})} \xrightarrow[\partial]{} \underset{(k\text{-chains})}{C_k(\mathcal{U})}.$$

Say $\omega \in A^k(\mathcal{U})$ and $\gamma \in C_{k+1}(\mathcal{U})$.

Goursat was the first to write down the multi-variable Fundamental Thm. of Calculus (a.k.a. Stokes's Theorem) in the form

(*) $$\int_\gamma d\omega = \int_{\partial\gamma} \omega .$$

Anyhow, going back to the area integral, the point is that working in $C^\infty_c(\mathcal{U}) \otimes \Lambda^2 \mathbb{R}^2$,
(or 1)

we have

$$du \wedge dv = (u_x\, dx + u_y\, dy) \wedge (v_x\, dx + v_y\, dy)$$

$$= u_x v_x \cancelto{0}{dx \wedge dx} + u_x v_y\, dx \wedge dy + u_y v_x \underbrace{dy \wedge dx}_{= -dx \wedge dy} + u_y v_y \cancelto{0}{dy \wedge dy}$$

since $dy \wedge dy = -dy \wedge dy$

$$= (u_x v_y - u_y v_x)\, dx \wedge dy .$$

(There is a tight relationship between determinants and top exterior powers of a vector space.)

The best place to first get acquainted with differential forms is "blue" Rudin.