Lecture 13: Complex integration

I. Paths

Let $S \subseteq \mathbb{C}$ be a subset.

"of class C^0"

A (parametrized) path in S is a continuous map

$$\gamma : [a,b] \longrightarrow S.$$

Context will dictate whether "γ" refers to the path or its image.

A partition Π of $[a,b]$ is a finite set $\{t_0, \dots, t_n\} \subset \mathbb{R}$ with

$$a = t_0 < t_1 < \cdots < t_n = b \quad ;$$

set $|\Pi| := \sup |t_i - t_{i-1}|$. The path γ is said to be rectifiable $\Longleftrightarrow$ its length

$$(*) \qquad L(\gamma) := \operatorname*{lub}_{\Pi} \sum_{i=1}^{n} \Big| \underbrace{\gamma(t_i) - \gamma(t_{i-1})}_{=:(\Delta\gamma)_i} \Big| \quad \text{is finite.}$$

Equivalently, we require

$(**)$ $\quad \operatorname{lub}_{\pi} \sum \left| \begin{cases} \mathrm{Re} \\ \mathrm{Im} \end{cases} \gamma(t_i) - \begin{cases} \mathrm{Re} \\ \mathrm{Im} \end{cases} \gamma(t_{i-1}) \right|$ both finite.

Clearly also

$$(*) \Leftarrow \underbrace{\lim_{|\pi| \to 0} \sum_{i=1}^{n} f(\gamma(t_i)) \, |(\Delta\gamma)_i|}_{=: \int_\gamma f \, ds} \text{ is defined for every } f \in C^0(\mathcal{S})$$

$$(**) \Leftrightarrow \underbrace{\lim_{|\pi| \to 0} \sum_{i=1}^{n} f(\gamma(t_i)) \, (\Delta\gamma)_i}_{=: \int_\gamma f \, dz} \text{ is defined for every } f \in C^0(\mathcal{S}).$$

Next, a (parametrized) curve in $\mathcal{S}$ is a continuously differentiable (class C^1) map

$$\gamma : [a, b] \to \mathcal{S}.$$

(γ is sometimes called regular if the derivative is nowhere vanishing.) To define the derivative, we can either write

$$\gamma(t) = x(t) + i y(t) \rightsquigarrow \gamma'(t) = x'(t) + i y'(t)$$

OR

$$\lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h} =: \gamma'(t).$$

This satisfies sum/product/quotient and chain rules:

for both

$$[c,d] \xrightarrow{\psi} [a,b] \xrightarrow{\gamma} \mathcal{S}$$

Δ in parametrization

AND

$$[a,b] \xrightarrow[\gamma]{} \mathcal{S} \xrightarrow[f]{} \mathbb{C}.$$

In the latter case

- $f \in C^1(\mathcal{S}) \Longrightarrow (f \circ \gamma)' = f_x x' + f_y y' = \frac{\partial f}{\partial z}\gamma' + \frac{\partial f}{\partial \bar{z}}\bar{\gamma}'$
- $f \in \mathcal{H}ol(\mathcal{S}) \Longrightarrow (f \circ \gamma)' = (f' \circ \gamma) \cdot \gamma'$

(that is, f has an extension to some region $U \supset \mathcal{S}$ which is C^1 resp. holo.)

A C^1 path (in $\mathcal{S}$) $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a collection of curves $\gamma_j : [a_j, b_j] \to \mathcal{S}$ satisfying $\gamma_{j+1}(a_{j+1}) = \gamma_j(b_j)$ $(j = 1, \ldots, n-1)$. (These are the objects over which we shall be interested in integrating.) Clearly

$$\gamma \text{ is } C^1 \Longrightarrow \gamma \text{ is rectifiable.}$$

If $\gamma, \tilde{\gamma} : [-\epsilon, \epsilon] \to D_1$ are curves with $\gamma(0) = 0 = \tilde{\gamma}(0)$ and $\gamma'(0), \tilde{\gamma}'(0) \neq 0$, and

$f \in \mathrm{Hol}(D_1)$ with $f'(0) \neq 0$, then

$$\frac{(f\circ\tilde{\gamma})'(0)}{(f\circ\gamma)'(0)} = \frac{\cancel{f'(0)}\,\tilde{\gamma}'(0)}{\cancel{f'(0)}\,\gamma'(0)} = \tilde{\gamma}'(0)/\gamma'(0) \implies$$

$$\arg \tilde{\gamma}'(0) - \arg \gamma'(0) = \arg (f\circ\tilde{\gamma})'(0) - \arg (f\circ\gamma)'(0)$$

which expresses conformality of f. On the other hand, if (say) $f(z) = z^3$ and $\gamma(t) = t$, $\tilde{\gamma}(t) = e^{i\theta}t$, then

$$(f\circ\gamma)(u^{1/3}) = u, \quad (f\circ\tilde{\gamma})(u^{1/3}) = e^{i3\theta}u.$$

We'll need some terminology related to global behavior of paths:

- γ is closed $\iff$ $\gamma(a) = \gamma(b)$
- γ is simple $\iff$ $\gamma|_{[a,b)}$ is 1-to-1

Jordan curve theorem γ simple & closed $\implies \mathbb{C}\setminus\gamma$ has 2 connected components.

II. Integration

Start with $\mathbb{C}$-valued functions on an interval

$$[a=t_0,\ t_1,\ t_2,\ \ldots\ldots,\ t_{N-1},\ b=t_N] \xrightarrow[\text{(continuous)}]{G} \mathbb{C}\ ,$$

$$G(t) = U(t) + i\,V(t).$$

Define $\displaystyle\int_a^b G\,dt := \int_a^b U\,dt + i\int_a^b V\,dt$

OR (equiv.) $\displaystyle := \lim_{N\to\infty} \frac{b-a}{N} \sum_{k=0}^{N-1} G(t_k)$ $\quad \left(t_k = a + k\frac{b-a}{N}\right)$

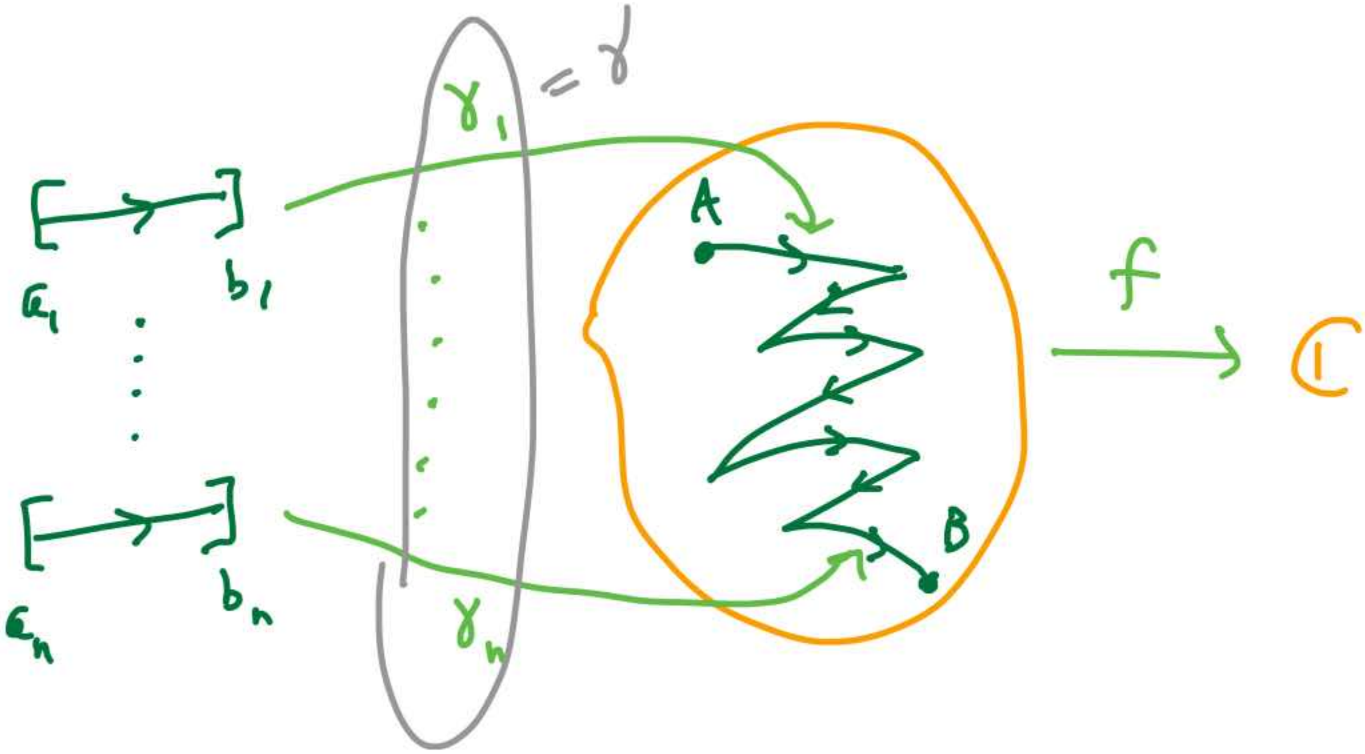
<u>How to bound</u>: $\displaystyle\left|\int_a^b G\,dt\right| = \lim_{N\to\infty} \frac{b-a}{N}\left|\sum G(t_k)\right|$

$$\leq \lim_{N\to\infty} \frac{b-a}{N} \sum |G(t_k)|$$

$$= \int_a^b |G|\,dt.$$

Things like integration by parts, fundamental theorem of calculus, and linearity (w/ ex. numbers) hold verbatim.

Now for a C^1 path $\gamma \subset U$ (= open + connected, say) and any $\underset{\approx}{f \in C^0(U)}$



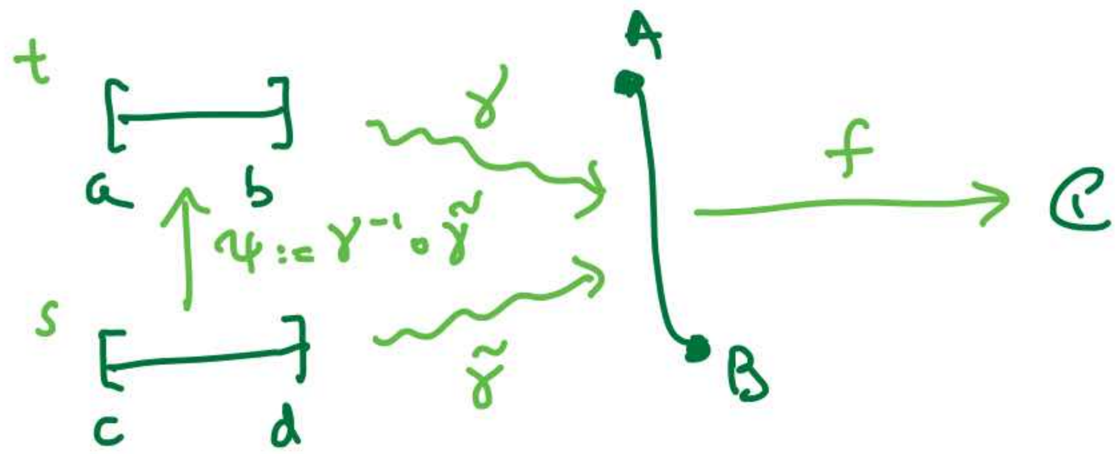
We define

$$\int_\gamma f(z)\,dz := \sum_i \underbrace{\int_{a_i}^{b_i} f(\gamma_i(t))\,\gamma_i'(t)\,dt}_{\int_{\gamma_i} f\,dz},$$

and write $\gamma = \sum \gamma_i$.

Remarks:

- this agrees w/ earlier def'n. for rectifiable γ
- $\int_\gamma (\alpha f + \beta g)\,dz = \alpha \int_\gamma f\,dz + \beta \int_\gamma g\,dz$
- "$-\gamma_i$" denotes the curve backwards: $\int_{-\gamma_i} f\,dz = -\int_{\gamma_i} f\,dz$
- Independence of parametrization: Suppose given two parametrizations of the same curve,



then $$\int_a^b f(\gamma(t))\, \gamma'(t)\, dt \underset{t = \psi(s)}{=} \int_c^d f(\underbrace{\gamma(\psi(s))}_{\tilde{\gamma}(s)})\, \underbrace{\gamma'(\psi(s))\, \psi'(s)}_{\tilde{\gamma}'(s)}\, ds$$

- Bounding integrals: put

$$L(\gamma) := \sum_i \underbrace{\int_{a_i}^{b_i} \overbrace{|\gamma_i'(t)|}^{\text{i.e. } \sqrt{(x')^2 + (y')^2}}\, dt}_{L(\gamma_i)} \left(= \int_\gamma ds \right) ;$$

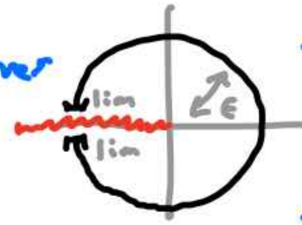
then

$$\left| \int_\gamma f(z)\, dz \right| \leq \sum_i \left| \int_{a_i}^{b_i} f(\gamma_i(t))\, \gamma_i'(t)\, dt \right|$$
$$\leq \sum_i \int_{a_i}^{b_i} \underbrace{|f(\gamma_i(t))|}_{\leq \|f\|_\gamma} |\gamma_i'(t)|\, dt$$
$$\leq \|f\|_\gamma \sum_i L(\gamma_i) = \|f\|_\gamma L(\gamma).$$

Example// Compute $\displaystyle\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} \log(z)\, dz$ $(= 0)$.

branch with $-\pi < \arg \leq \pi$

[<u>Note</u>: this has to be interpreted in the right way

— $\int_{\partial D_\epsilon}$ is a limit of integrals over curves 

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$$\left|\int_{\partial D_\epsilon} \log z \, dz\right| \le \|\log z\|_{\partial D_\epsilon} L(\partial D_\epsilon)$$

$$\le \left(|\log \epsilon| + \pi\right) 2\pi\epsilon \xrightarrow[\epsilon \to 0]{} 0 . \qquad //$$

- **FTC (Fund. Thm. of Calculus):**

If $f \in C^0(U)$ has primitive $F \in Hol(U)$ (i.e. $F' = f$)

then
$$\int_\gamma f(z)\,dz = \sum_i \int_{a_i}^{b_i} \underbrace{f(\gamma_i(t))\,\gamma_i'(t)}_{(F\circ\gamma_i)'(t)}\,dt$$

$$= \sum_i \left(F(\gamma_i(b_i)) - F(\gamma_i(a_i))\right)$$

$$= F(B) - F(A).$$

Collapsing sum

Now, if $g \in C^1(U)$ (and $\gamma \subset U$, $f \in C^0(U)$)

there's nothing preventing us from defining $\underline{\int_\gamma f \, dg}$.

Here you need to think of

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = \frac{\partial g}{\partial z}dz + \frac{\partial g}{\partial \bar z}d\bar z ,$$

and $\begin{cases} dz = \gamma'(t)\,dt \\ d\bar{z} = \overline{\gamma'}(t)\,dt \end{cases}$, $\begin{cases} dx = \mathrm{Re}\,\gamma'(t)\,dt \\ dy = \mathrm{Im}\,\gamma'(t)\,dt \end{cases}$.

This gives (assume γ is a curve)

$$\int_\gamma f\,dg = \int_a^b f(\gamma(t))\left\{\frac{\partial g}{\partial z}\gamma'(t) + \frac{\partial g}{\partial \bar{z}}\overline{\gamma'(t)}\right\}dt$$

$$= \int_a^b f(\gamma(t))\langle(\nabla g)(\gamma(t)), \gamma'(t)\rangle\,dt ,$$

dot product

and (assuming both f & g are C^1)

- Integration by parts: $\int_\gamma g\,df = fg\Big|_A^B - \int_\gamma f\,dg$.

III. Loop integrals

A loop is a closed path.

Proposition Given $U \subset \mathbb{C}$ region, $f \in C^0(U)$:

① f has a (holo.) primitive on U (i.e. $\exists\, F \in \mathrm{Hol}(U)$ s.t. $F' = f$)

$\Updownarrow$

② $\int_A^B f\,dz$ is independent of path ($\forall\, A, B \in U$)

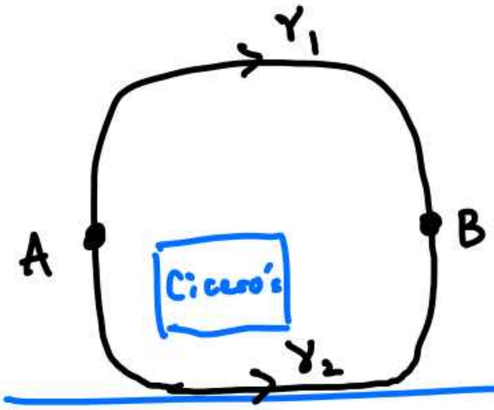
$\Updownarrow$

③ $\int_\gamma f\,dz = 0$ ($\forall\, C^1$ loops $\gamma \subset U$).

Remark: It's sort of amazing that you only need to assume $f \in C^0(U)$ to get (3) $\Rightarrow$ (1).

Proof: (3) $\Rightarrow$ (2): Set $\gamma := \gamma_2 - \gamma_1 \implies \gamma$ closed

$$\overset{(3)}{\implies} 0 = \int_\gamma f\,dz = \int_{\gamma_2} f\,dz - \int_{\gamma_1} f\,dz$$



Palmer

(2) $\Rightarrow$ (3): Break a given loop into $\gamma_2 - \gamma_1$, use indep. of path $\implies \int_\gamma f\,dz = 0$.

(1) $\Rightarrow$ (2): FTC

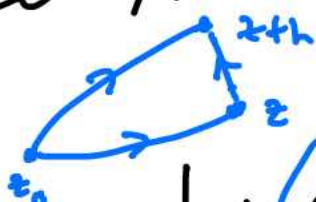
(2) $\Rightarrow$ (1): Define $F(z) := \int_{z_0}^{z} f(w)\,dw = \int_\gamma f(w)\,dw$

z_0 fixed $\in U$

for any γ connecting z_0 to z (by (2), choice doesn't matter, hence F is well-defined).

Must show F is holo. w/ f as derivative: show

$\lim_{h \to 0} = 0$ of



straight path

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \overset{(2)}{=} \left| \frac{1}{h} \int_z^{z+h} f(w)\,dw - f(z) \right|$$

$$= \left| \frac{1}{h}\int_z^{z+h} \cancel{\underset{\uparrow\,\text{const.}}{f(z)}}\,dw - \cancel{f(z)} + \frac{1}{h}\int_z^{z+h} \varepsilon(w)\,dw \right|$$

$f(w) = f(z) + \varepsilon(w)$

$\lim_{w \to z} \varepsilon(w) = 0$

(since $f \in C^0$)

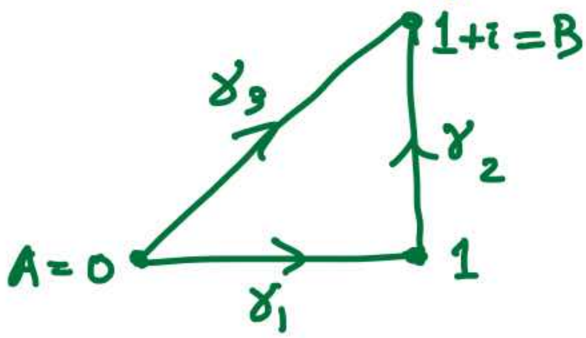
$$\le \left|\frac{1}{h}\right| \overbrace{|h|}^{L(\text{path})} \|\varepsilon(w)\|_{\underset{z,\, z+h}{\text{path}}} \longrightarrow 0 \quad \text{as } h \to 0. \qquad \Box$$

Basic examples // functions w/ holo. primitive:

- on $\mathbb{C}$: z^n $(n \ge 0)$, e^z, $\sin z$, $\cos z$
- on $\mathbb{C}^*$: z^n $(n \le -2)$
- on D_r : any function given by power series at 0 w/ radius of convergence r.

All have trivial loop-integrals about the origin. //

Basic non-example // $f(z) = \bar{z}$.



On $[0,1]$, $\begin{cases} \gamma_1(t) = t \\ \gamma_2(t) = 1 + it \\ \gamma_3(t) = (1+i)t \end{cases}$

$$\int_{\gamma_i} \bar{z}\,dz = \int_0^1 \overline{\gamma_i(t)}\ \underbrace{\gamma_i'(t)\,dt}_{dz}, \quad \text{so}$$

$$\int_{\gamma_1} \bar{z}\, dz = \int_0^1 t \cdot 1\, dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\int_{\gamma_2} \bar{z}\, dz = \int_0^1 (1-it)\cdot i\, dt = i\int_0^1 dt + \int_0^1 t\, dt = \frac{1}{2} + i.$$

$$\int_{\gamma_3} \bar{z}\, dz = \int_0^1 (1-i)t \cdot (1+i)\, dt = t^2\Big|_0^1 = 1.$$

$$\Rightarrow \int_{\gamma_1+\gamma_2} \bar{z}\, dz = \int_{\gamma_1} \bar{z}\, dz + \int_{\gamma_2} \bar{z}\, dz = \left(i+\frac{1}{2}\right) + \frac{1}{2} = 1+i$$

$$\neq 1 = \int_{\gamma_3} \bar{z}\, dz.$$

//

Lemma Given $\sum_{k=0}^{n} f_k \xrightarrow{} f$ uniformly on $\gamma \supseteq \gamma$, C^0

$$\sum_{k=0}^{n}\left(\int_\gamma f_k(z)\, dz\right) = \int_\gamma \underbrace{\left(\sum_0^\infty f_k\right)}_{\shortparallel} dz.$$

Proof:

$$\left|\sum_0^n \int_\gamma f_k\, dz - \int_\gamma f\, dz\right| = \left|\int_\gamma \left(\sum_0^n f_k\right) dz - \int_\gamma f\, dz\right|$$

$$= \left|\int_\gamma \left(\sum_0^n f_k - f\right) dz\right| \leq L(\gamma)\left\|\sum_0^n f_k - f\right\|_\gamma$$

$$\xrightarrow[n\to\infty]{} 0$$

since the series uniformly converges on γ. $\square$

Proposition Let f be given on $S := \overline{D}_r \setminus \{0\}$ by the Laurent series $(f(z) :=)\ \frac{a_{-m}}{z^m} + \dots + \frac{a_{-1}}{z} + \underbrace{\sum_{n=0}^{\infty} a_k z^k}_{\text{w/ radius of conv.} > r}$.

Then $\int_{\partial D_r} f(z)\,dz = 2\pi i\, a_{-1}$.

Proof: $\sum_0^\infty$ converges uniformly on $\overline{D}_r$ to a continuous function, so (writing $\gamma = \partial D_r$)

$$\int_\gamma \sum_0^\infty a_k z^k\,dz = \sum_0^\infty a_k \int_\gamma z^k\,dz = 0$$

since $\begin{cases} \gamma \ \underline{\text{closed}} \\ z^k \text{ has primitive } \frac{z^{k+1}}{k+1} \text{ on } D_{r+\epsilon}. \end{cases}$

For $m \neq -1$, the other $\int$'s are also 0; but

$$\int_\gamma \frac{1}{z}\,dz = \int_0^1 \frac{1}{\cancel{re^{2\pi i t}}} \underbrace{2\pi i \cancel{r} \cancel{e^{2\pi i t}}}_{\gamma'(t)}\,dt = 2\pi i \int_0^1 dt$$

$\begin{pmatrix} z = \gamma(t) := re^{2\pi i t} \\ t \in [0,1] \end{pmatrix}$

$$= 2\pi i.$$

$\square$