Lecture 12: Constructing conformal equivalencies (CE)

When you see the Riemann mapping theorem later on, you'll get more sophisticated methods to do this, simply by considering the map along the boundary.

For now, the tools are (for constructing or studying maps)

- FLTs. These give CEs from $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

  It's also easy to "see" what these do since they take $\mathcal{C} \to \mathcal{C}$.

- powers. $z \mapsto z^\alpha = e^{\alpha \log z}$ takes

$$\underbrace{S(\varphi_1, \varphi_2)}_{0 < \varphi_2 - \varphi_1 \leq 2\pi} := \{ z \in \mathbb{C}^* \mid \varphi_1 < \arg z < \varphi_2 \}$$

to "$S(\alpha\varphi_1, \alpha\varphi_2)$",

which makes sense expressed this way (and is 1-1) iff $0 < \alpha(\varphi_2 - \varphi_1) \leq 2\pi$. $\left[e^{i\varphi} \mapsto e^{\alpha \log(e^{i\varphi})} = e^{i\alpha\varphi} \right]$

- **level curves**. We know that

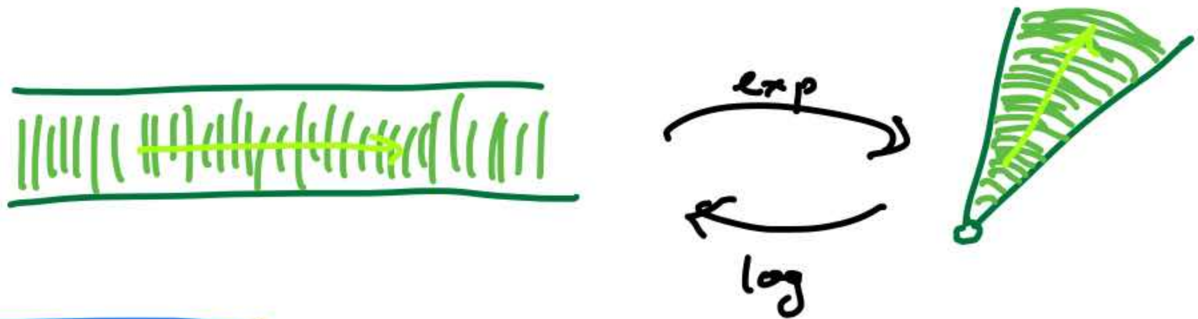
$F(x=x_0)$ and $F(y=y_0)$ are orthogonal
(in the image)

$F^{-1}(u=u_0)$ and $F^{-1}(v=v_0)$ are orthogonal
(in the domain)

(or use polar coordinates).

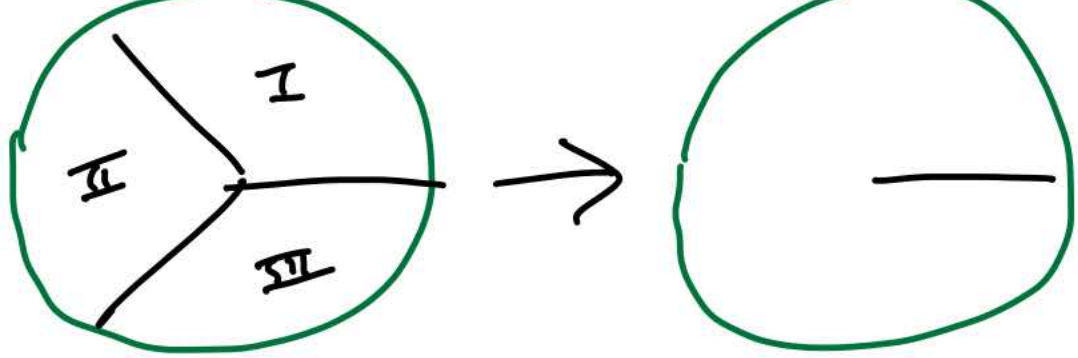
- **exp & log**. $z \longmapsto e^z$ maps strips to sectors:

$$\{x+iy \mid y \in (\varphi_1, \varphi_2)\} \underset{1-1}{\longleftrightarrow} S(\varphi_1, \varphi_2)$$



EXAMPLES

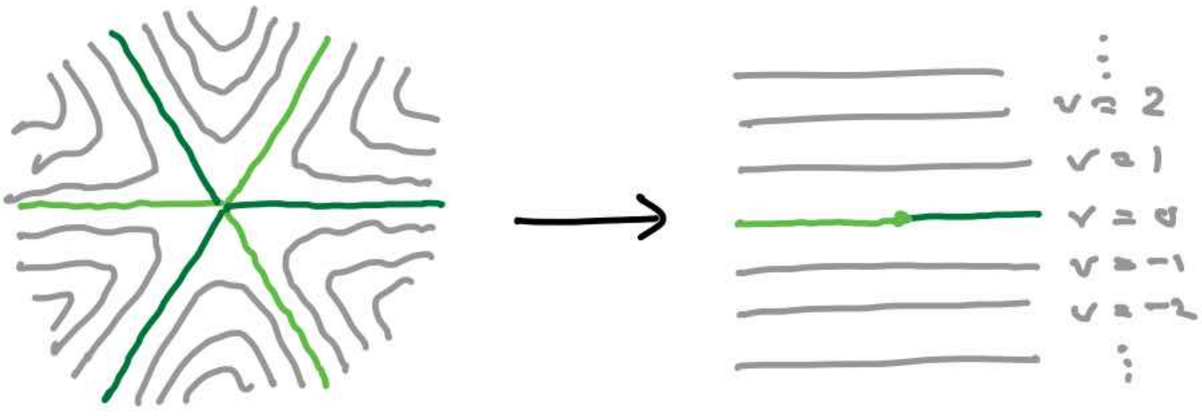
① $z \to z^3$. gives CE of "I, II, III" with the slit disk:



preimage of "$v = v_0$" is given by

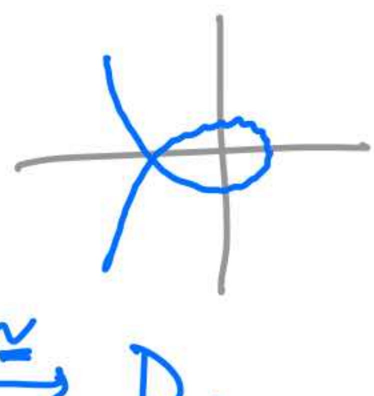
$$v_0 = \mathrm{Im}\left((x+iy)^3\right) = 3yx^2 - y^3 = y(3x^2-y^2).$$

If $v_0 = 0$, this is $y/x = 0, \sqrt{3}$, or $-\sqrt{3}$.



Rotating the left-hand figure by $30°$ gives the preimage of the vertical lines. (The curves in the rotated figure are $\perp$ to those in this one.)

By drawing a (say) vertical line on top of the figure showing the level curves, you "see" that $x = x_0 > 0$ gets sent to



② $F : \hat{\mathbb{C}} \setminus [-1,-1] \xrightarrow{\cong} D_1$

$$
\begin{tikzcd}
U = \hat{\mathbb{C}} \setminus [-1,1] \arrow[d, "\cong"', "\text{1-1 since FLT}"] & z \arrow[d, mapsto] \\
\mathbb{C} \setminus \mathbb{R}_{\geq 0} \arrow[d, "\cong"', "\sqrt{\cdot}"] & \frac{1+z}{1-z} \arrow[d, mapsto] \\
h \arrow[d, "\cong"', "\text{use } v \mapsto \frac{v-i}{v+i}"] & \sqrt{\frac{1+z}{1-z}} \arrow[d, mapsto] \\
D_1 & \frac{\sqrt{1+z} - i\sqrt{1-z}}{\sqrt{1+z} + i\sqrt{1-z}}
\end{tikzcd}
$$

$$
\frac{\sqrt{1+z} - i\sqrt{1-z}}{\sqrt{1+z} + i\sqrt{1-z}} = \frac{\left(\sqrt{1+z} - i\sqrt{1-z}\right)^2}{1+z+1-z}
$$

$$
= \frac{1}{2}\left(1+z-(1-z) - 2i\sqrt{1-z^2}\right)
$$

$$
= z - i\sqrt{1-z^2}
$$

$$
= z - \sqrt{z^2-1}
$$

$$
=: F(z) \quad (= w)
$$

Solve for inverse: $\quad z - w = \sqrt{z^2-1}$

$$
z^2 - 2zw + w^2 = z^2 - 1
$$

$$
(*) \qquad \boxed{z = \frac{1}{2}\left(w + \frac{1}{w}\right)}
$$

and we can "pull back" under this inverse map to see where a given curve goes under F.

Consider the set of (confocal) ellipses in the z-plane with foci ± 1:

$$|z+1| + |z-1| = C \quad (> 2)$$

square

$$2|z|^2 + \underbrace{2|z^2-1|}_{\text{i.e. } \frac{1}{2}|4z^2-4|} = C^2 - 2$$

plug in (*)

$$\frac{1}{2}\left(w+\frac{1}{w}\right)\left(\bar{w}+\frac{1}{\bar{w}}\right) + \frac{1}{2}\underbrace{\left|w^2+w^{-2}-2\right|}_{\left|w-\frac{1}{w}\right|^2 = \left(w-\frac{1}{w}\right)\left(\bar{w}-\frac{1}{\bar{w}}\right)} = C^2-2$$

$$|w|^2 + \frac{1}{|w|^2} = C^2 - 2$$

$$|w|^4 + (2-C^2)|w|^2 + 1 = 0$$

$$|w|^2 = \frac{(C^2-2) \pm \sqrt{C^4-4C^2}}{2} = \frac{C^2-2-C\sqrt{C^2-4}}{2}$$

choose $\ominus$ root so that $|w|<1$

$$|w| = \sqrt{\frac{C^2}{2}\left(1-\sqrt{1-\frac{4}{C^2}}\right)-1}$$

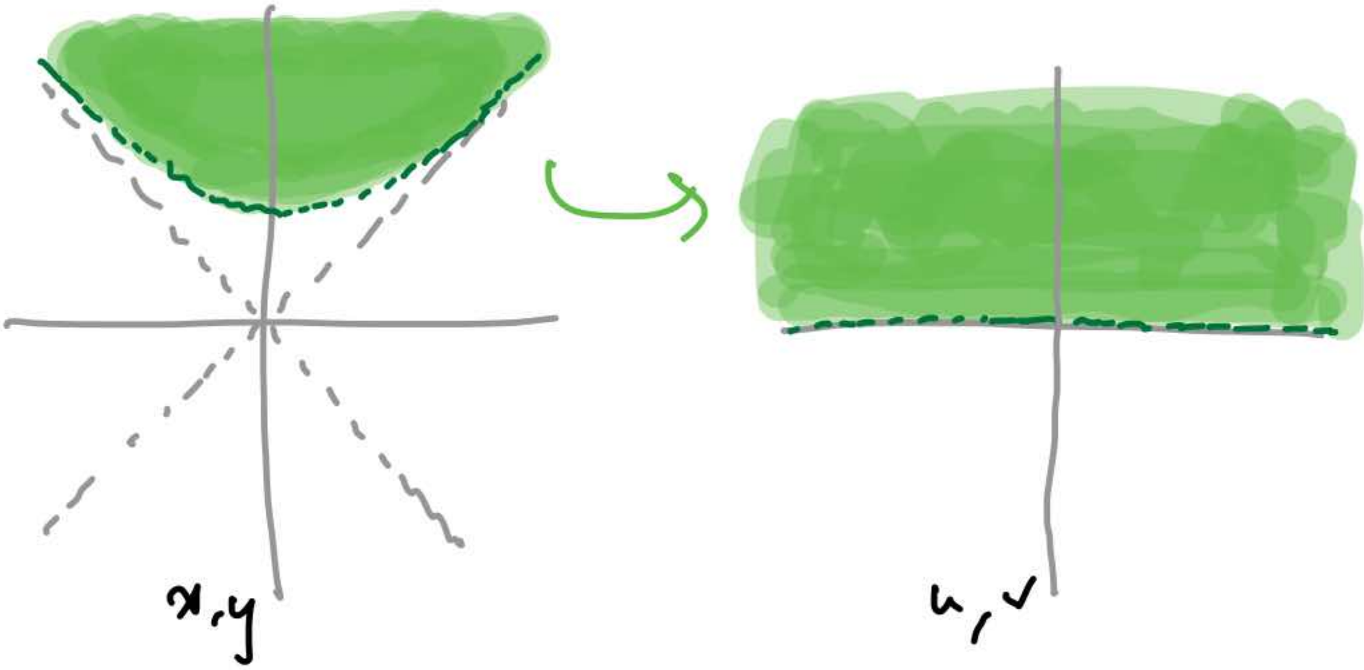
$$=: R(C)$$

$\left(\approx \frac{1}{C^2}\right.$ for C large.

For C small $\to 2^+$, $\left. R(C) \to 1^- \right)$

i.e. they map to circles!

③ $R = \{y^2 > x^2 + 1\} \xrightarrow{\cong} h \ \left(\cdots \xrightarrow{\cong} D_1 \right)$



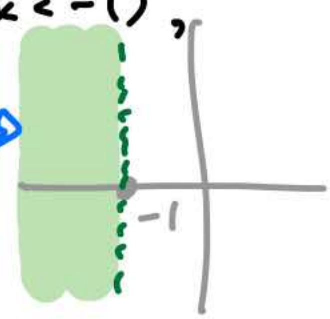
Well, $z \longmapsto z^2$ would at least open up the $\frac{1}{4}$-plane R sits in, to a $\frac{1}{2}$-plane!

But look:

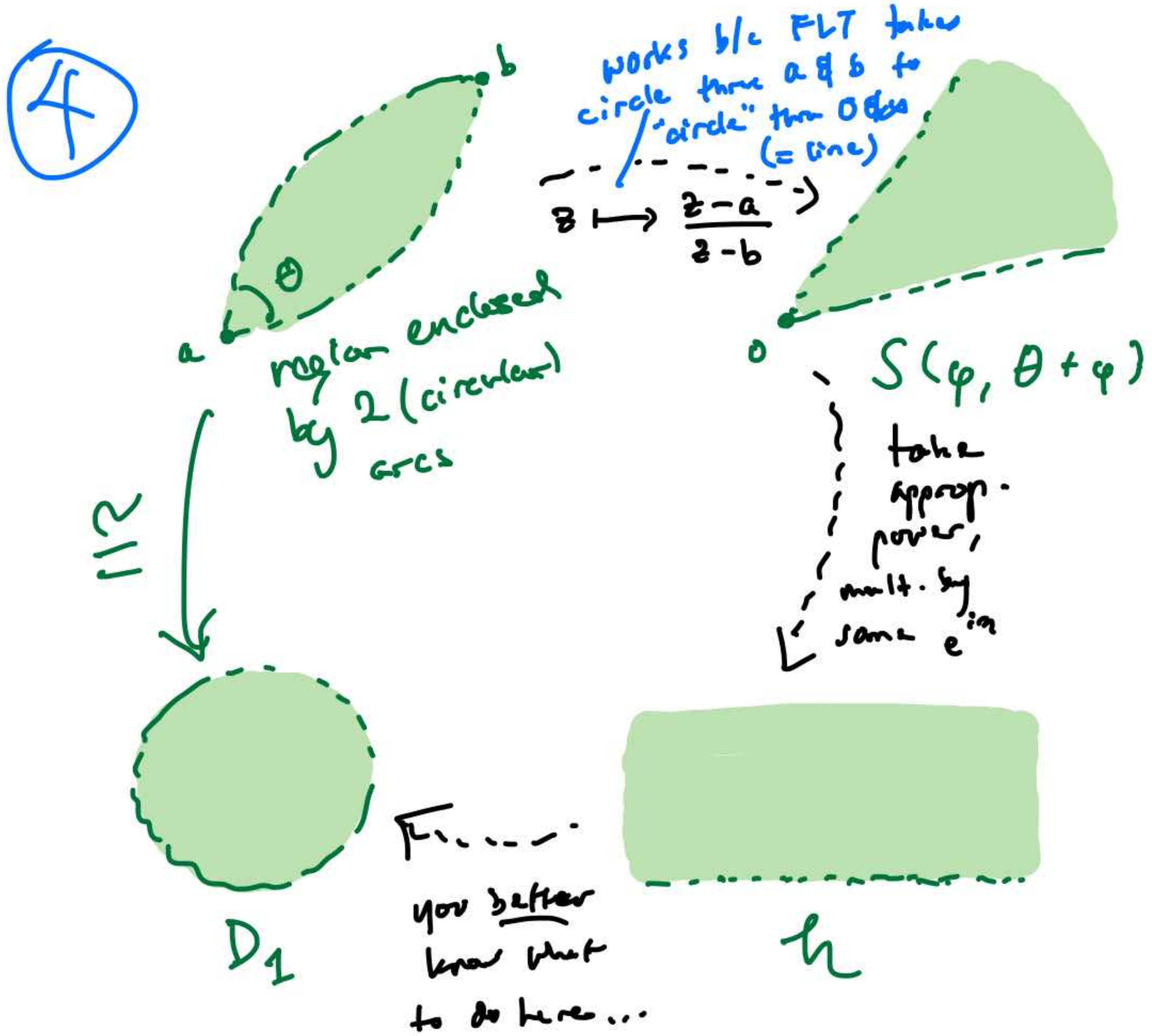
$$z^2 = (x+iy)^2 = (x^2-y^2) + i(2xy) = u + iv$$

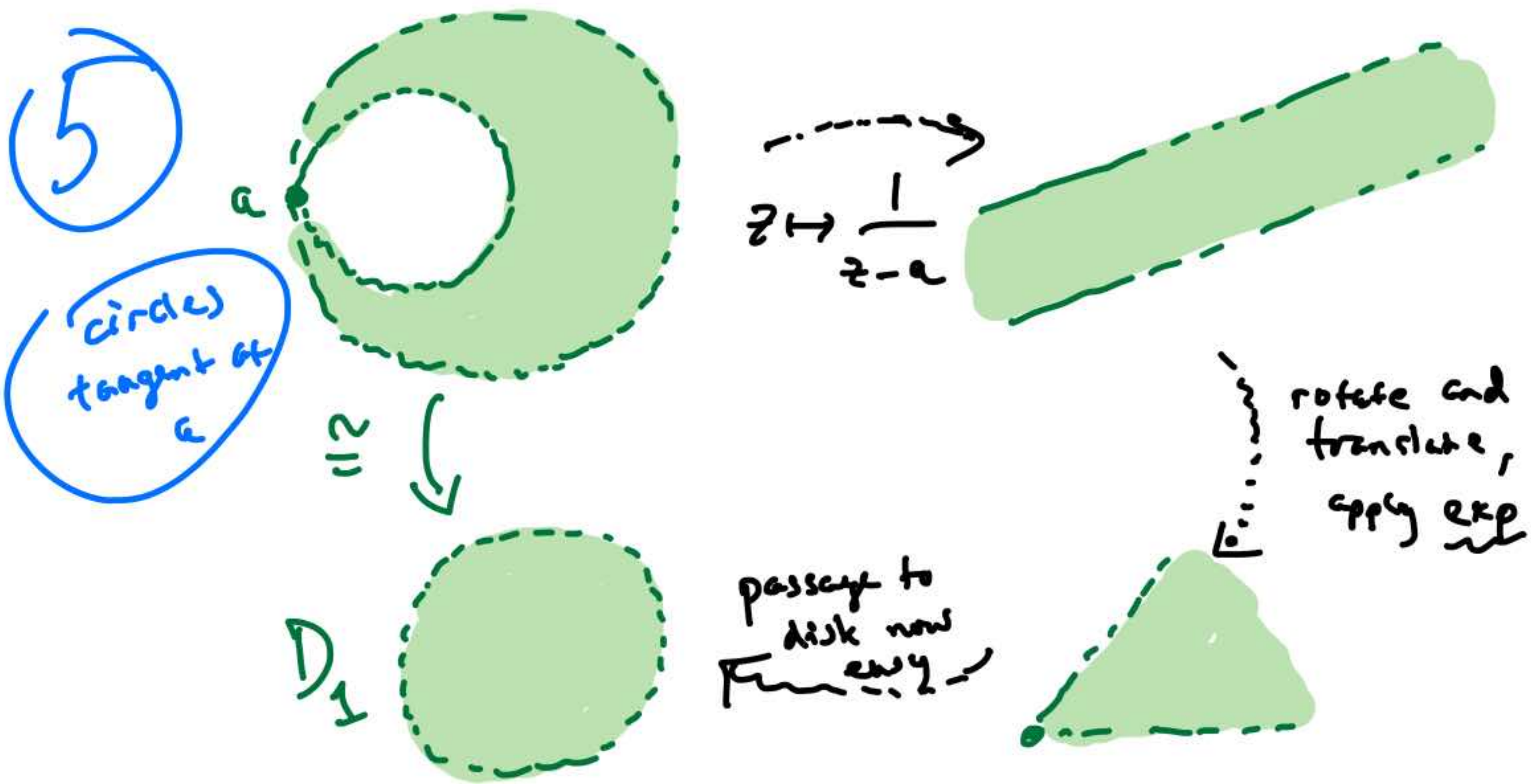
$$\Longrightarrow x^2 - y^2 = -k \quad (\text{i.e. } y^2 = x^2 + k) \text{ goes to } u = -k.$$

Since R is filled out by the hyperbolas $y^2 = x^2 + k$ for $k > 1$ (by definition) and these are mapped in 1-1 fashion onto lines $u = -k$ (for $-k < -1$), all we then have to do is add 1 and multiply by $-i$: so we get

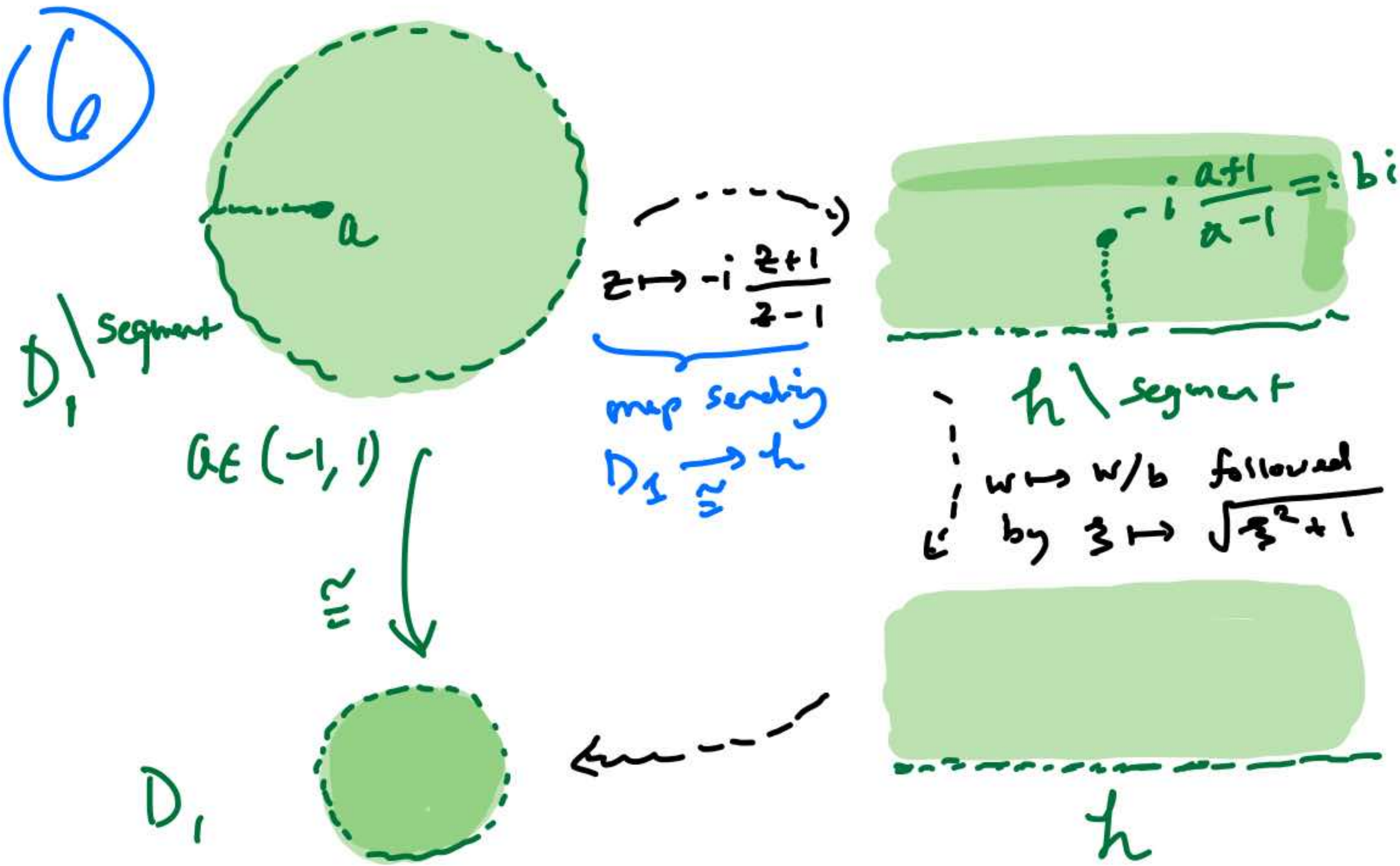


$$z \longmapsto -i(z^2+1).$$


4
b
works b/c FLT takes circle thru a & b to "circle" thru 0 & ∞ (= line)
$z \mapsto \frac{z-a}{z-b}$
θ
a
region enclosed by 2 (circular) arcs
0
$S(\varphi, \theta+\varphi)$
$\cong$
take approp. power, mult. by some $e^{i\alpha}$
D_1
you better know what to do here...
$\mathbb{H}$


5
a
circles tangent at a
$z \mapsto \frac{1}{z-a}$
rotate and translate, apply exp
$\cong$
D_1
passage to disk now easy

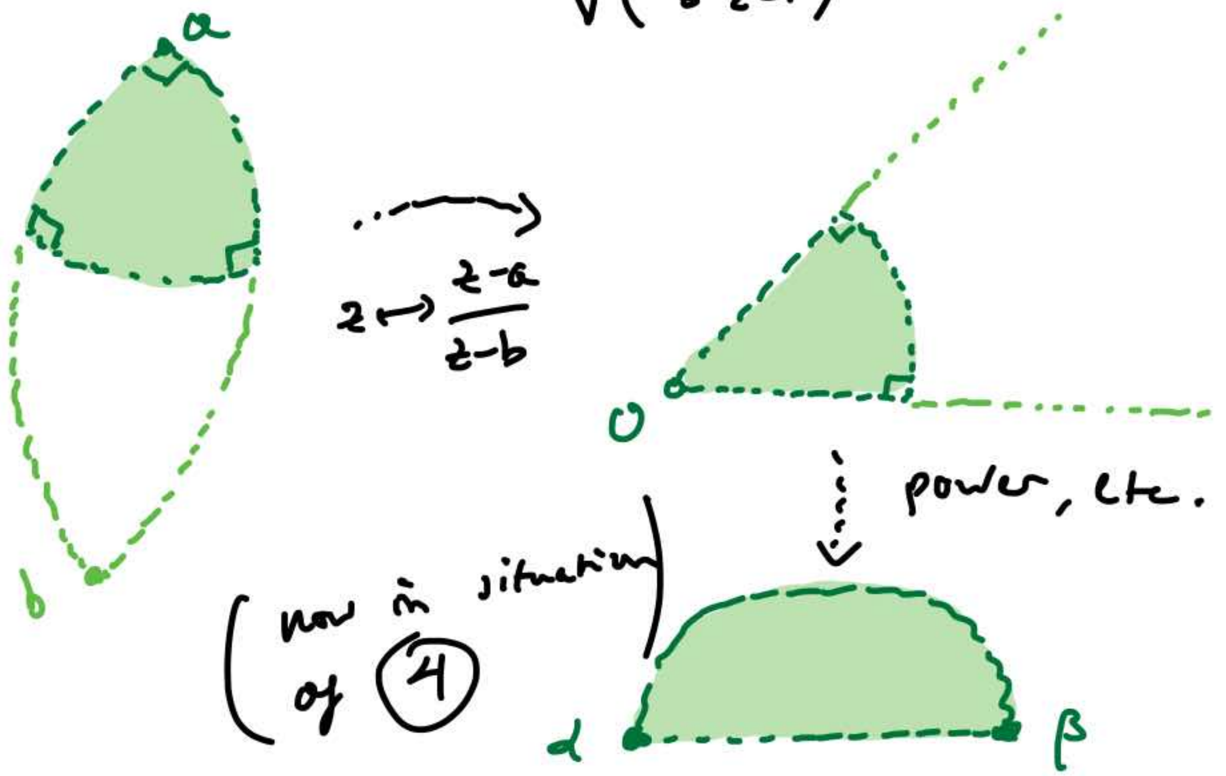
(6)



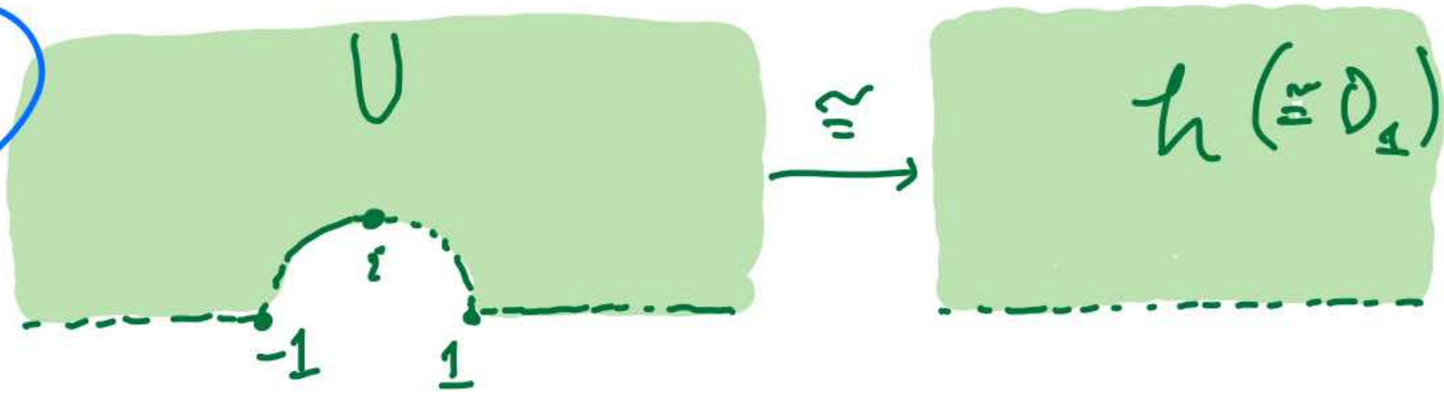
composite map :

$$z \longmapsto \frac{\sqrt{\left(-\frac{i}{b}\frac{z+1}{z-1}\right)^2+1} - i}{\sqrt{\left(-\frac{i}{b}\frac{z+1}{z-1}\right)^2+1} + i}.$$

(7)



(8)



$$z \longmapsto z + \frac{1}{z} \left(= w = f(z) \right)$$

Proof:

$$w = x + iy + \frac{x - iy}{(x+iy)(x-iy)}$$

$$= x\left(1 + \frac{1}{x^2+y^2}\right) + iy\left(1 - \frac{1}{x^2+y^2}\right)$$

"into"

$z \in U \Rightarrow w \in h$:

$$\left.\begin{array}{l} |z| > 1 \\ \mathrm{Im}(z) > 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x^2+y^2 > 1 \\ y > 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \frac{1}{x^2+y^2} < 1 \\ y > 0 \end{array}\right\}$$

$$\Rightarrow \mathrm{Im}(w) = y \cdot \left(1 - \frac{1}{x^2+y^2}\right) > 0.$$

"onto"

given $w \in h$, find solution of $w = z + \frac{1}{z}$ in U:

$$wz = z^2 + 1$$

$$0 = z^2 - wz + 1 = (z - z_+)(z - z_-)$$

Clearly $z_+ z_- = 1$; we can't have $|z_+|$ & $|z_-| = 1$, because then $z_- = \overline{z_+} \Rightarrow (z - z_+)(z - z_-) = z^2 - \underbrace{(z_+ + \overline{z_+})}_{w = 2\mathrm{Re}(z) \notin \mathfrak{h}} z + 1$.

So let $|z_+| > 1$, $|z_-| < 1$; then

$$w = z_+ + \frac{1}{z_+} = z_+ + z_- \in \mathfrak{h} \Rightarrow \mathrm{Im}(z_+) > 0,$$

which together with $|z_+| > 1 \Rightarrow z_+ \in U$.

(-1) Use fact that $|z_-| < 1$ so not in U. □