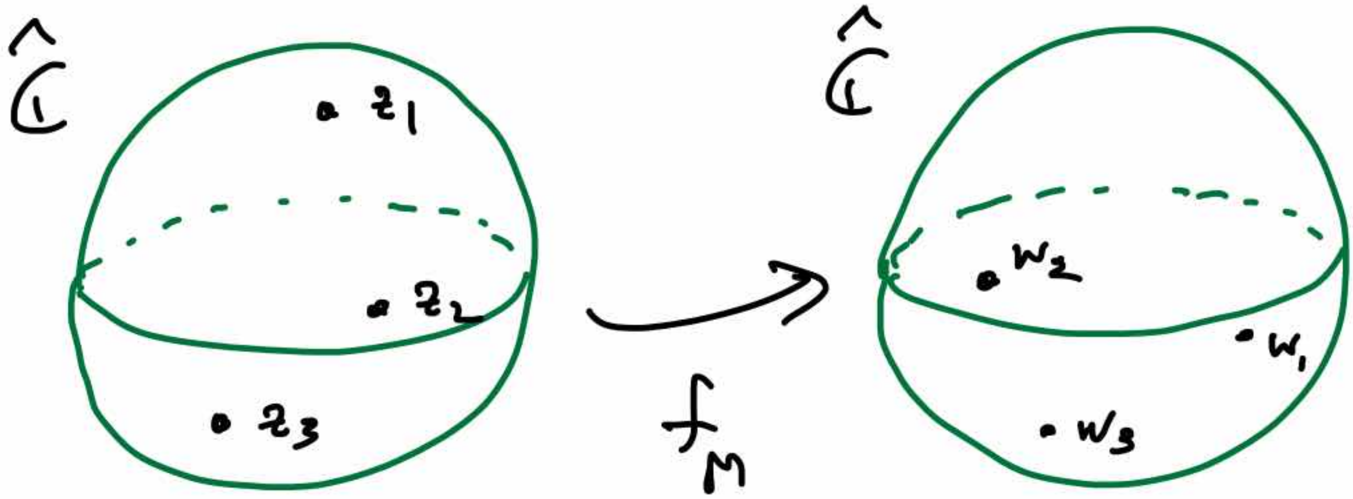
Lecture 11: Conformal Mappings

We continue our discussion of FLTs (= fractional linear transformations), then consider a generalization.

I. Cross-ratio



Recall that there exists a unique $f_M \in \underline{FLT}$ taking $z_i \longmapsto w_i$ $(\forall i)$. In the special case where $\{w_i\}$ are $1, 0, \infty$, this is

$$\boxed{f(z) = \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2} =: CR(z, z_1, z_2, z_3)}$$

which tells us where z is sent by the FLT taking the last 3 entries to $1, 0, \infty$ (in that order).

$\boxed{\text{Theorem}}$ (a) CR is invariant under FLTs: if $g \in \underline{FLT}$, then

(*) $$CR(g(z), g(z_1), g(z_2), g(z_3)) = CR(z, z_1, z_2, z_3).$$

(b) z, z_1, z_2, z_3 belong to the same $C \in \mathcal{C}$ $\iff$ $CR(z, z_1, z_2, z_3) \in \mathbb{R}$.

Proof: (a) $f \circ g^{-1}$ sends $g(z_1), g(z_2), g(z_3) \mapsto 1, 0, \infty$.

So $LHS(*) = (f \circ g^{-1})(g(z)) = f(z) = RHS(*)$.

(b) ($\Longrightarrow$): If z lies on $C := \underbrace{C_{(z_1, z_2, z_3)}}_{\text{(cf. Corollary from Lect. 10)}}$ then (by transitivity of action of $\underline{FLT}$ on $\mathcal{C}$) $\exists$ $F \in \underline{FLT}$ taking C into $\mathbb{R}$. But then

$$CR(z, z_1, z_2, z_3) = CR(F(z), F(z_1), F(z_2), F(z_3)) \in \mathbb{R}.$$

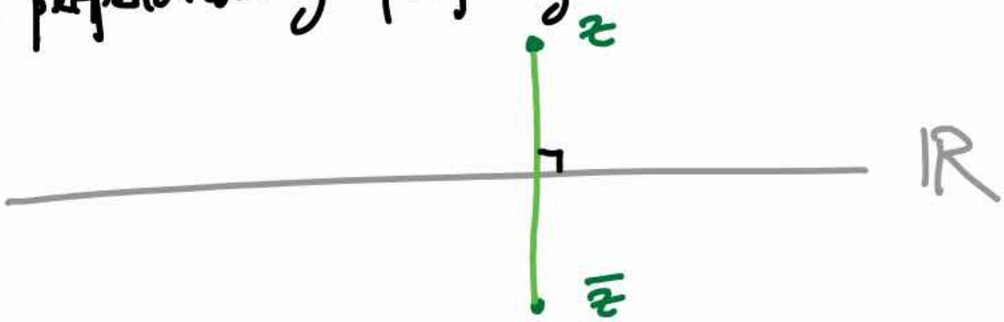
($\Leftarrow$): Let $G \in \underline{FLT}$ send $1, 0, \infty$ to z_1, z_2, z_3 hence $\mathbb{R}$ to $C := C_{(z_1, z_2, z_3)}$.

Then $G^{-1}(z) = CR(G^{-1}(z), \underbrace{1, 0, \infty}_{G^{-1}(z_1),\, G^{-1}(z_2),\, G^{-1}(z_3)}) = CR(z, z_1, z_2, z_3) \in \mathbb{R}$

$\Rightarrow z = G(G^{-1}(z)) \in G(\mathbb{R}) = C$. $\square$

II. Symmetry and orientation

Complex conjugation is a reflection about $\mathbb{R}$, with the "perpendicularity property" shown in the picture:



Are there corresponding reflections about arbitrary $C \in \mathcal{C}$?

Let $F \in \underline{FLT}$ take $\mathbb{R} \to C$, & define

$$z^*_C := F\left(\overline{F^{-1}(z)}\right) = (F \circ \overline{F}^{-1})(\bar{z}).$$

(Here the bar denotes complex conjugation, and $\overline{F}$ means $F_{\overline{M}}$ if $F = F_M$.) Given any other $\tilde{F} \in \underline{FLT}$

taking $\mathbb{R} \to C$,

$$F^{-1} \circ \tilde{F} = f_M, \quad M \in SL_2(\mathbb{R}).$$

So $\tilde{F} = F \circ f_M \implies \tilde{F}(\bar{\tilde{F}}^{-1}(\bar{z}))$

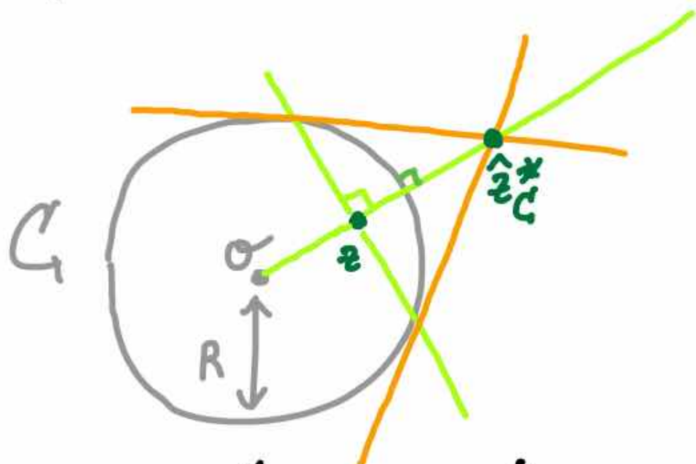
$$= F(f_M(\bar{f}_M^{-1}(\bar{F}^{-1}(\bar{z}))))$$

$$= F(\bar{F}^{-1}(\bar{z}))$$

$\bar{f}_M = f_M$

and z^*_C is well-defined.

The above picture holds for any line, since we can take F to be a composite of rotation/dilations and translations. For any line or circle, $(\cdot)^*_C$ fixes points on C. Now consider Ahlfors's construction



We will show that $\hat{z}^*_C = z^*_C$. First, we claim that $d(z, \sigma) \cdot d(z^*_C, \sigma) = R$. [It suffices to check everything for $C = \{|z| = 1\}$ (since we can translate the center to 0 and dilate), where

$R=1$. The FLT $z \longmapsto \frac{z-i}{z+i} = F(z)$ sends $\mathbb{R} \to \partial D_1$, and has $F^{-1}(z) = \frac{i(1+z)}{1-z}$, so

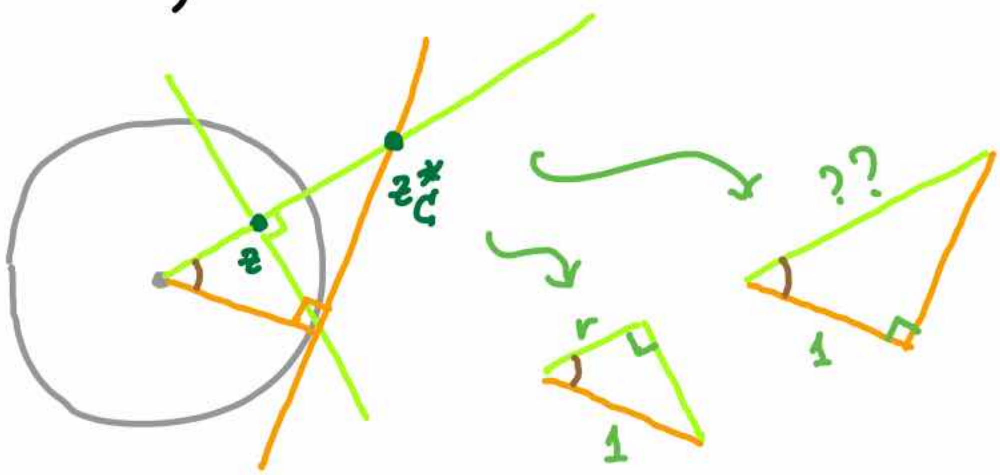
$$F\left(\overline{F^{-1}(z)}\right) = \frac{-\frac{i(1+\bar{z})}{1-\bar{z}} - i}{-\frac{i(1+\bar{z})}{1-\bar{z}} + i} = \dots = \frac{1}{\bar{z}} \quad ;$$

that is,

$$(re^{i\theta})^*_{\partial D_1} = \frac{1}{re^{-i\theta}} = \frac{1}{r}e^{i\theta},$$

confirming the claim.]

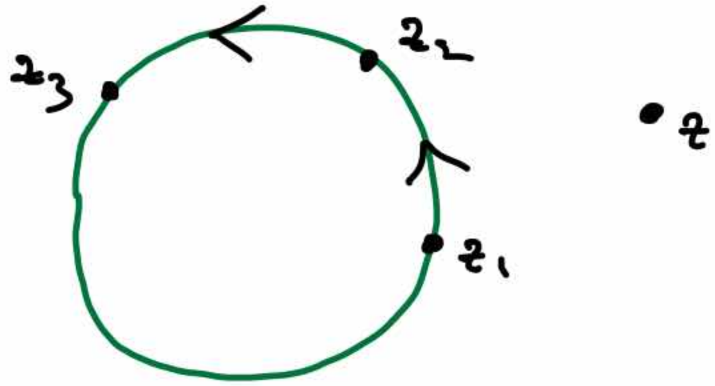
Now taking $z = re^{i\theta}$, consider the similar triangles



Clearly the "??" edge has length $\frac{1}{r}$, and so

$$\hat{z}^*_{\partial D_1} = \frac{1}{r}e^{i\theta} \underset{\text{from above}}{=} z^*_{\partial D_1}$$

as promised. The geometric picture now makes it clear that the segment connecting z^*_C and z meets C orthogonally. Moreover, they lie on different "sides" of C. If we use the ordering of z_1, z_2, z_3 on C to give it an orientation



then

- z is to the right of the (oriented) circle $\overset{\text{def.}}{\iff}$

$$\operatorname{Im}\left(CR(z, z_1, z_2, z_3)\right) > 0$$

- z is to the left of the (oriented) circle $\overset{\text{def}}{\iff}$

$$\operatorname{Im}\left(CR(z, z_1, z_2, z_3)\right) < 0 .$$

The circle is oriented counterclockwise if ∞ is to the right.

III. Fixed points

Let $f(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$. A fixed point of f is a solution to the equation

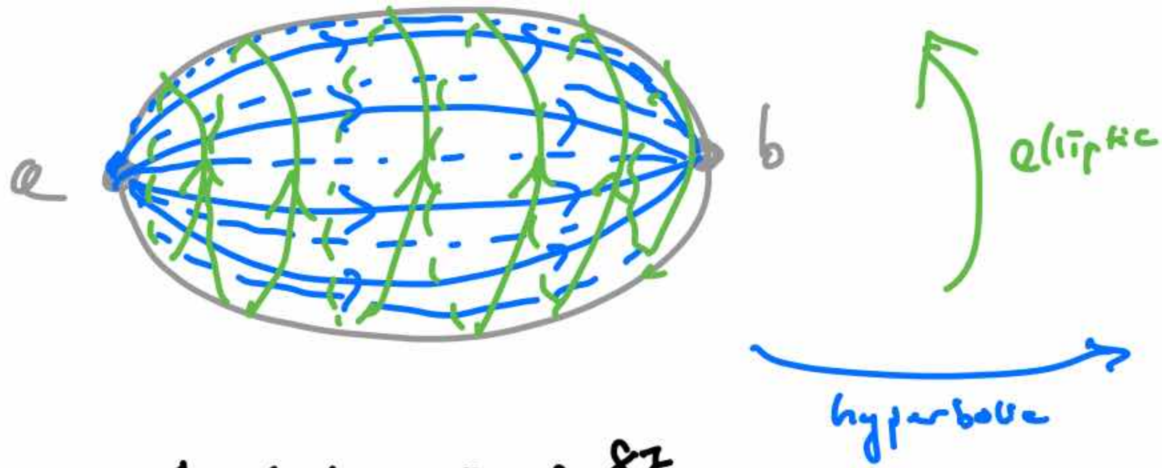
$$z = \frac{\alpha z + \beta}{\gamma z + \delta}$$

or (equiv.) $\gamma z^2 + (\delta - \alpha) z - \beta = 0$

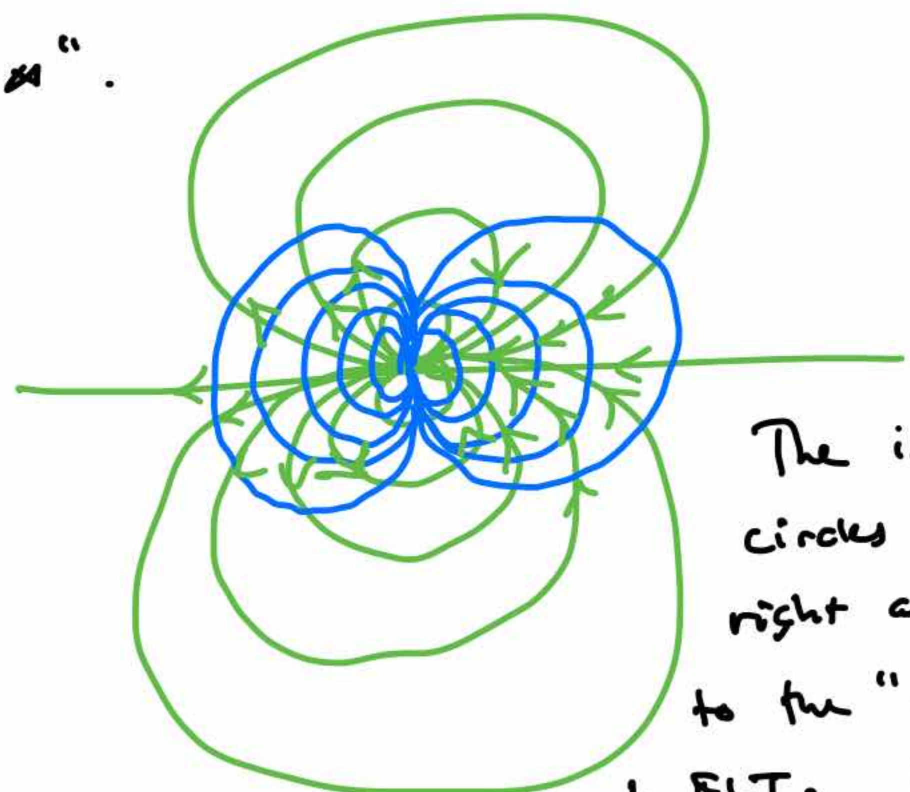
$$z = \frac{\alpha - \delta \pm \sqrt{(\delta-\alpha)^2 + 4\gamma\beta}}{2\gamma}.$$

- If $\gamma = 0$ then one or both FP's (counted w/ multiplicity) are at ∞. In this case, f is "affine" or "loxodromic."
- If $\gamma \neq 0$ and $(\delta - \alpha)^2 = -4\gamma\beta$ then there is one (finite) FP with multiplicity 2, and f is "parabolic."
- If there are 2 distinct finite roots, then call them a & b, and take σ to be an FLT sending $a, b \mapsto 0, \infty$ (e.g. $\sigma(z) = \frac{z-a}{z-b}$). Then $\sigma \circ f \circ \sigma^{-1}$ fixes 0 & ∞. (Note that this simply diagonalizes the corresponding matrix.) The FLTs fixing 0 & ∞ are just the μ_ξ ($\xi \in \mathbb{C}^*$): dilation-rotations. So $\sigma \circ f \circ \sigma^{-1} = \mu_\xi$. If $\xi \in \mathbb{R}^*$ then f is "hyperbolic"; if $\xi \in S^1$ "elliptic"; otherwise (again) "loxodromic".

Remark: (a) The pictures of circles on p. 85 of Ahlfors are just the image under $z \mapsto \frac{z-a}{z-b}$ (sends $0, \infty \mapsto a, b$) of polar coordinate lines. The meaning is clearer on $\hat{\mathbb{C}}$:



(b) The pictures of conics on p. 87 are just "a view of Cartesian coordinates from ∞", and the transformations he describes are just "translations seen from ∞".



The intersecting circles meet at right angles due to the "conformality" of FLTs. (See next §).

(c) The transformation

$$\frac{(\cos\theta)z + \sin\theta}{-(\sin\theta)z + \cos\theta}$$

fixes i & $-i$. Diagonalizing the corresponding matrix, or just computing the characteristic polynomial, you find the eigenvalues are $\cos\theta \pm i\sin\theta$, i.e. $|\xi| = 1$

$\Rightarrow$ elliptic. (In general, ξ is just the ratio of the eigenvalues.)

In Iwasawa, $N \leftrightarrow$ affine (or parabolic if you count ∞ as a legitimate FP)

$A \leftrightarrow$ affine (or hyperbolic if - - - - - - - - - - - - -)

$K \leftrightarrow$ elliptic.

(d) In fact, I would propose the following (revised) terminology: working mod $\pm id$, with $M \in SL_2(\mathbb{C})$,

f_M elliptic $\Leftrightarrow$ M conjugate to $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ $(\theta \in \mathbb{R})$

f_M hyperbolic $\Leftrightarrow$ M conjugate to $\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$ $(r \in \mathbb{R}_{>0})$

f_M parabolic $\Leftrightarrow$ M conjugate to $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ $(\alpha \in \mathbb{C})$

Then every FLT is a product of elliptic, hyperbolic, and parabolic transformations, due to the Jordan decomposition.

IV. Conformal mappings : generalities

Let's first nail down a definition.

Let $U, V \subset \mathbb{C}$ be regions, and $F : U \to V$ a mapping with continuous partial derivatives.

Definition F is conformal on $S \underset{\text{open}}{\subset} U \overset{\text{def.}}{\iff}$

$F|_S$ preserves angles to first order $\underset{\text{algebra}}{\overset{\text{linear}}{\iff}}$

($J_F(z)$ is of form $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ $(\forall z \in S)$ $\overset{\text{Lecture 2}}{\iff}$

$F|_S$ is holomorphic with nonvanishing derivative.)

How do we know a holomorphic (or analytic) map $f : U \to V$ is conformal?

I mean a 2-sided inverse here.

Proposition If $\exists$ holomorphic $g : V \to U$ inverting f, then f is conformal on all of U.

Proof: $1 = (g \circ f)'(z_0) = g'(f(z_0)) \cdot f'(z_0) \Rightarrow f'(z_0) \neq 0$. $\square$

(It will turn out that 1-1 is enough.)

In this situation, f is said to be a holomorphic isomorphism $U \xrightarrow[\cong]{f} V$, and also (because of the Proposition) a conformal equivalence.

When are 2 open sets in $\mathbb{C}$ conformally equivalent? Here is a famous result:

$\boxed{\text{Riemann Mapping Theorem}}$ Given $U \subsetneq \mathbb{C}$ a simply connected region. Then there exists a conformal equivalence $f : U \xrightarrow{\cong} D_1$.

Example $U = \mathfrak{h}$, $f(z) := \frac{z-i}{z+i}$.

Claim: $\mathfrak{h} \xrightarrow[\cong]{f} D_1$.

Pf: $f(\mathfrak{h}) \subseteq D_1$: $z \in \mathfrak{h} \Rightarrow z = x + iy$, $y > 0$.

$$
\begin{aligned}
y > 0 &\Rightarrow y^2 - 2y + 1 < y^2 + 2y + 1 \\
&\Rightarrow (y-1)^2 < (y+1)^2 \\
&\Rightarrow x^2 + (y-1)^2 < x^2 + (y+1)^2 \\
&\Rightarrow |z - i|^2 < |z + i|^2 \\
&\Rightarrow |z - i| < |z + i| \\
&\Rightarrow |f(z)| < 1 \\
&\Rightarrow f(z) \in D_1 .
\end{aligned}
$$

Note: that f is 1-1 follows from the fact that it is a FLT (& has holo. inverse)

$f(h) \supseteq D_1$: $w \in D_1 \Rightarrow \operatorname{Im}\left(-i \frac{w+1}{w-1}\right)$

$$
\begin{aligned}
&= \operatorname{Re}\left(-\frac{w+1}{w-1}\right) \\
&= \operatorname{Re}\left(-\frac{(w+1)(\bar{w}-1)}{|w-1|^2}\right) \\
&= \operatorname{Re}\left(-\frac{|w|^2 - 1 + \bar{w} - w}{|w-1|^2}\right) \\
&= \frac{1 - |w|^2}{|w-1|^2} > 0
\end{aligned}
$$

$$\Rightarrow -i\frac{w+1}{w-1} \in h .$$

Now $f\left(-i\frac{w+1}{w-1}\right) = \dfrac{-i\frac{w+1}{w-1} - i}{-i\frac{w+1}{w-1} + i} = \dfrac{w+1+(w-1)}{w+1-(w-1)}$

$= w$, done. $\square$

Picture: