Lecture 10: Fractional linear transformations

I. Group structure

In this course we'll meet the automorphism groups (of 1-to-1 analytic self-maps)

$$\mathrm{Aut}(\mathbb{C}) = \{ z \mapsto \alpha z + \beta \mid \alpha, \beta \in \mathbb{C},\ \alpha \neq 0 \}$$

$$\mathrm{Aut}(D_1) = \left\{ z \mapsto e^{i\varphi} \frac{z-\alpha}{1-\bar{\alpha} z} \,\middle|\, \varphi \in \mathbb{R},\ \alpha \in D_1 \right\}$$

$$\mathrm{Aut}(\mathfrak{h}) = \left\{ z \mapsto \frac{az+b}{cz+d} \,\middle|\, a,b,c,d \in \mathbb{R},\ ad-bc = 1 \right\}$$

as well as the isomorphism

$$\mathfrak{h} \xrightarrow{\cong} D_1$$

$$z \longmapsto \frac{z-i}{z+i}.$$

All are fractional linear transformations (FLT)

$$f_M(z) := \frac{az+b}{cz+d}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \underbrace{GL_2(\mathbb{C})}_{\substack{a,b,c,d \in \mathbb{C} \\ ad-bc \neq 0}}.$$

If we consider the FLT's as a group under composition of functions, then

$$M \longmapsto f_M(z)$$

defines a surjective group homomorphism

$$GL_2(\mathbb{C}) \longrightarrow \underline{FLT}:$$

- if $N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$,

$$(f_M \circ f_N)(z) = \frac{a\left(\frac{Az+B}{Cz+D}\right)+b}{c\left(\frac{Az+B}{Cz+D}\right)+d} = \cdots$$

$$= \frac{(aA+bC)z + (aB+bD)}{(cA+dC)z + (cB+dD)}$$

$$= f_{M\cdot N}(z).$$

- $f_M(z) = z$ (identity) $\Longleftrightarrow$ $M = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ $(\alpha \in \mathbb{C}^*)$:

$$\rightarrow \quad \frac{\alpha z + 0}{0z + \alpha} = z$$

$\rightarrow$ if $\frac{az+b}{cz+d} = z$ $(\forall z \in \mathbb{C})$, then $0 = cz^2 + (d-a)z + b$ $(\forall z)$

$$\Rightarrow c = (d-a) = b = 0$$

$$\Rightarrow M = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

So in fact our homomorphism factors through

$$PGL_2(\mathbb{C}) := GL_2(\mathbb{C}) \Big/ \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \,\middle|\, \alpha \in \mathbb{C}^* \right\} \xrightarrow{\cong} \underline{FLT}.$$

$$\left(\cong SL_2(\mathbb{C}) \big/ \{\pm id\} \right)$$

But what are the FLTs really transformations *of*?

We can view f_M as a function from

$$\begin{array}{lccc} & \mathbb{C} \setminus \{-d/c\} & \xrightarrow{\cong} & \mathbb{C} \setminus \{a/c\} \\ & \cap| & & \cap| \\ \text{which may be extended to} & \hat{\mathbb{C}} & \xrightarrow{\cong} & \hat{\mathbb{C}} \\ \text{by setting} & -d/c & \longmapsto & \infty \\ & \infty & \longmapsto & a/c. \end{array}$$

Remark// In fact, if we define $\mathrm{Aut}(\hat{\mathbb{C}})$ by

- $f \in \mathrm{Aut}(\hat{\mathbb{C}})$ means (i) $f : \mathbb{C}\setminus\{\alpha\} \xrightarrow{\cong} \mathbb{C}\setminus\{\beta\}$

  (ii) $1/f : U(\alpha) \xrightarrow{\cong} U(0)$

  (iii) $f(1/w) : U(0) \xrightarrow{\cong} U(\beta)$

  "U" means a small neighborhood

  analytic

then

- $\exists\ f^{-1}$ with similar properties (for any such f)

and

- $\boxed{\underline{\text{FLT}} \cong \text{Aut}(\hat{\mathbb{C}}).}$ (*)

Heuristic sketch of (*): That FLT $\hookrightarrow$ Aut$(\hat{\mathbb{C}})$ is easy. Now let f be an arbitrary $\mathbb{C}$-analytic automorphism of $\hat{\mathbb{C}}$: automorphism $\Rightarrow$ 1-to-1 $\Rightarrow$ no essential singularities $\Rightarrow$ no common limit points for zeroes & poles.

By our previous results, we can therefore have no limit points of zeroes or poles of f (otherwise f or $\frac{1}{f}$ is identically zero). Since $\hat{\mathbb{C}}$ is compact, there are therefore only finitely many zeroes and poles; multiplying by a rational function gets rid of these, leaving us with an analytic map $\hat{\mathbb{C}} \to \mathbb{C}$. Since (again) $\hat{\mathbb{C}}$ is compact, this is bounded, hence by Liouville constant. We conclude that f was rational ($= P/Q$, P & Q polynomials). But then, removing any common factors, the mapping degree of f is the maximum of $\deg(P)$ & $\deg(Q)$. This must be 1 for f to be 1-to-1. So P & Q are constant or linear $\Longrightarrow f \in$ FLT. $\square$

This used "everything", including Casorati–Weierstraß, Fundamental Thm. of Algebra, Liouville, etc.!! Mathematica for what still has to be proved, I guess ...

So we have

$$SL_2(\mathbb{C}) \big/ \{\pm \mathrm{id}\} \cong \underline{\mathrm{FLT}} \cong \mathrm{Aut}(\hat{\mathbb{C}}).$$

(or $PGL_2(\mathbb{C})$) (unofficially)

The group structure on $\underline{\mathrm{FLT}}$ clarifies lots of stuff:

- Composition inverses: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ (using $ad-bc=1$)

$$\Longrightarrow f^{-1}_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(w) = \frac{dw-b}{-cw+a}.$$

Indeed, that $\underline{\mathrm{FLT}}$ is a group (under composition) means that all FLTs are 1-to-1.

- Iwasawa decomposition: for real FLTs (more on these below)

one has $$SL_2(\mathbb{R}) \cong N \cdot A \cdot K$$

$$\text{any } M \in SL_2(\mathbb{R}) \overset{!}{=} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

- <u>Why 2×2 matrices?</u> If you think of $\hat{\mathbb{C}}$ as

$$\mathbb{P}^1 = \frac{\mathbb{C}^2 \setminus \{(0,0)\}}{\langle (z_1, z_2) \sim (\alpha z_1, \alpha z_2) \rangle_{\alpha \in \mathbb{C}^*}} \underset{\longleftarrow}{\overset{\cong}{\longrightarrow}} \hat{\mathbb{C}},$$

$$\left[\begin{array}{rcl} [z:1] & \longleftarrow\!\!\shortmid & z \neq \infty \\ (z_2 \neq 0) \quad [z_1 : z_2] & \longmapsto & z_1/z_2 \\ [1:0] & \longleftarrow\!\!\shortmid & \infty \\ [z_1 : 0] & \longmapsto & \infty \end{array}\right]$$

and write elements $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ instead of $[z_1 : z_2]$, then

[in $\mathbb{P}^1$] $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}$$

$$\updownarrow \qquad\qquad \updownarrow$$

[in $\hat{\mathbb{C}}$] $$z \left(= \frac{z_1}{z_2}\right) \longmapsto \frac{az_1 + bz_2}{cz_1 + dz_2} = \frac{a\left(z_1/z_2\right) + b}{c\left(z_1/z_2\right) + d}$$

$$= \frac{az+b}{cz+d}$$

So really, FLTs are linear transformations acting on lines <u>in $\mathbb{C}^2$</u> through the origin.

II. Action on circles

Let

$\mathcal{C}$:= set of circles & lines on $\mathbb{C}$

= set of circles on $\hat{\mathbb{C}}$

(viewed as a sphere via stereographic projection)

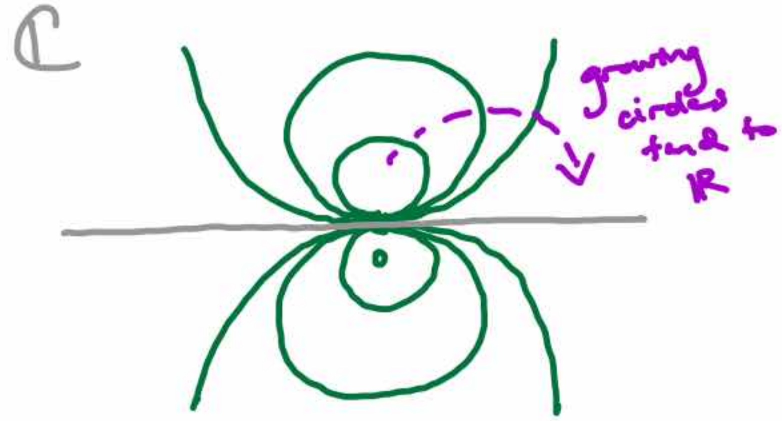
(see end of Lecture 1)

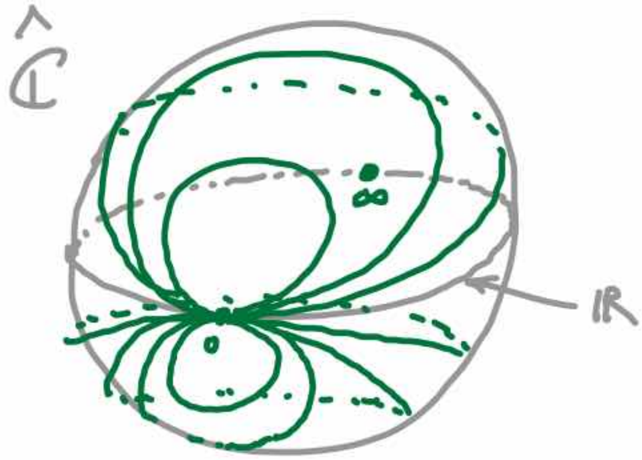
Theorem (a) $f \in FLT$ takes $\mathcal{C} \to \mathcal{C}$

(b) This action is transitive.

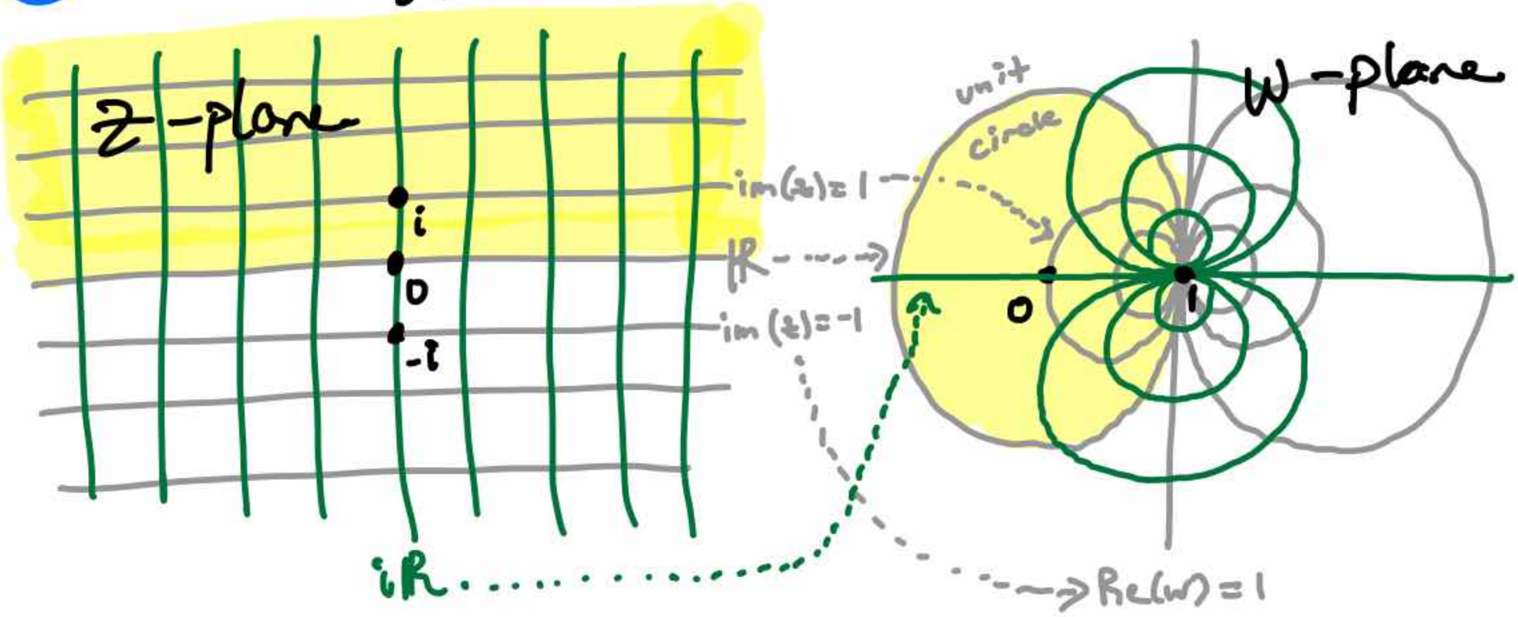
Examples

① lines are special kinds of circles: ones that contain $\{\infty\}$.



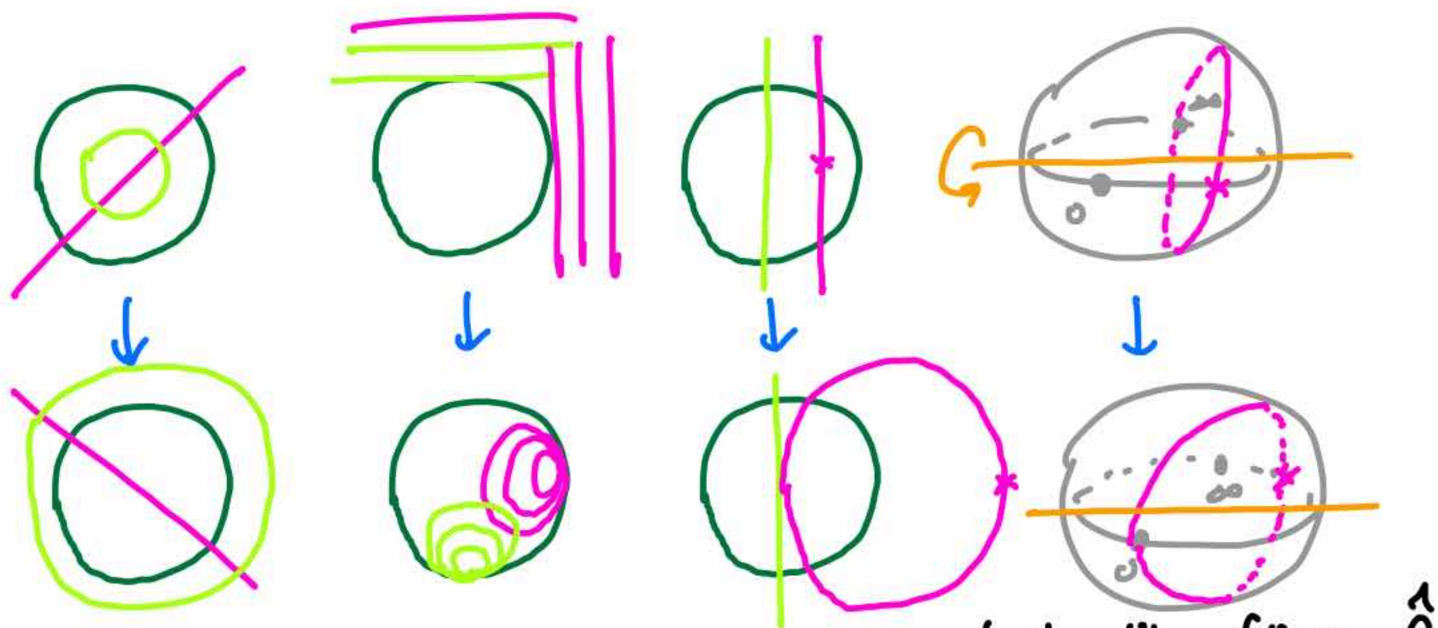


② $z \longmapsto \frac{z-i}{z+i} =w$ (takes $h \xrightarrow[\cong]{} D_1$)



In some sense, the last picture ("degenerate Steiner circles") is telling you "what Cartesian coordinates look like at ∞" (about $w=1$).

③ $z \longmapsto \frac{1}{z} =: J(z)$ (inversion)



looks like flipping $\hat{\mathbb{C}}$ 180° about the orange line

④ $f(z) = \dfrac{az+b}{cz+d}$ with $a,b,c,d \in \mathbb{R}$ sends $\mathbb{R} \to \mathbb{R}$ (and these are the only FLTs doing this); if $ad-bc>0$, then also sends $\mathfrak{h} \xrightarrow[\cong]{} \mathfrak{h}$. These are automorphisms of $\hat{\mathbb{C}}$ which (in the  picture) don't mix the upper & lower hemispheres.

The N·A·K decomposition certainly suggests that they preserve circles: translation by x; dilation by y; only issue is what the "K" matrices/FLTs do. //

Proof of (a): $F(z) = \dfrac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{C}$.

$$c = 0 \Rightarrow F(z) = \frac{a}{d}z + \frac{b}{d} = \tau_{b/d} \circ \mu_{a/d}.$$

translation by element in $\mathbb{C}$

multiplication by element in $\mathbb{C}^*$

$$c \neq 0 \Rightarrow F(z) = \frac{az + \frac{d}{c}a}{cz+d} + \frac{b - \frac{d}{c}a}{cz+d}$$

$$= \frac{a}{c}\,\underbrace{\cancel{\frac{cz+d}{cz+d}}}_{1} + \frac{b/c - da/c^2}{z + d/c}$$

$$= \tau_{a/c} \circ \mu_{\frac{bc-da}{c^2}} \circ J \circ \tau_{d/c} \,.$$

Now, τ & μ preserve $\mathcal{C}$; how about J?

(Then F is a composition of such things; done.)

$$J(x+iy) = u+iv \Rightarrow J(u+iv) = x+iy$$

$$J(u+iv) = \frac{1}{u+iv} = \underbrace{\frac{u}{u^2+v^2}}_{x} + i\underbrace{\frac{-v}{u^2+v^2}}_{y}$$

Hence, given an object

$$C = \{A(x^2+y^2) + Bx + Cy + D = 0\}$$

of $\mathcal{C}$, we can rewrite the equation in u,v to get

$$J(C) = \left\{A\left(\frac{u^2+v^2}{(u^2+v^2)^2}\right) + B\frac{u}{u^2+v^2} - C\frac{v}{u^2+v^2} + D = 0\right\}$$

$$= \{A + Bu - Cv + D(u^2+v^2) = 0\}$$

which is again in $\mathcal{C}$. $\square$

To prove (b), we'll need a

$\boxed{\text{Lemma}}$ Given $\begin{cases} z_1, z_2, z_3 \text{ distinct} \\ w_1, w_2, w_3 \text{ distinct} \end{cases} \left(\in \hat{\mathbb{C}}\right)$,

$\exists!\ f_M \in \underline{\text{FLT}}$ sending $z_i \mapsto w_i$ $(i=1,2,3)$.

Proof of $\exists$: First, send w's/z's to $0, 1, \infty$:

$$f(z) = \frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1}, \quad g(w) = \frac{w-w_1}{w-w_3} \cdot \frac{w_2-w_3}{w_2-w_1}$$

$$z\text{'s} \overset{f}{\longmapsto} 0,1,\infty \overset{g}{\longmapsfrom} w\text{'s}$$

$\Rightarrow$

$$(g^{-1} \circ f)(z_i) = w_i \quad (\forall i).$$

so take $w = g^{-1}(f(z))$, i.e. $g(w) = f(z)$. $\square$

Example// Find f_M sending $\overset{z_1\ z_2\ z_3}{-1, 0, 1} \mapsto \overset{w_1\ w_2\ w_3}{-1, i, 1}$.

set $\dfrac{z+1}{z-1} \cdot \dfrac{-1}{1} = \dfrac{w+1}{w-1} \cdot \underbrace{\dfrac{i-1}{i+1}}_{=i}$

$$(z+1)(w-1) = -i(w+1)(z-1)$$

$$zw + w - z - 1 = -iwz - iz + iw + i$$

$$\Rightarrow w = \frac{z+i}{iz+1} = f_M(z) \Rightarrow M = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

//

Proof of !: If f, g both map $\{z_i\} \mapsto \{w_i\}$,

then $g^{-1} \circ f$ fixes $\{z_i\}$. Let F send $\{z_i\}$ to $\{0, 1, \infty\}$. Then $h = F \circ g^{-1} \circ f \circ F^{-1}$ fixes $0, 1, \infty$

$$\Rightarrow \quad h(z) = \frac{az+b}{cz+d} \quad : \quad \infty \mapsto \infty \quad \Rightarrow \quad c = 0, \; d, a \neq 0$$

$$\Rightarrow \quad = Az + B \quad : \quad 0 \longmapsto 0 \quad \Rightarrow \quad B = 0$$

$$\Rightarrow \quad = Az \quad : \quad 1 \longmapsto 1 \quad \Rightarrow \quad A = 1$$

$$\Rightarrow \quad = z = id(z).$$

So $id = F \circ g^{-1} \circ f \circ F^{-1}$

$$F^{-1} \circ id \circ F = g^{-1} \circ f$$

$$id = g^{-1} \circ f$$

$$g = f.$$

$\square$

Corollary $\exists !$ $C \in \mathcal{C}$ through any 3 distinct $z_i \in \mathbb{C}$.

Proof: ($\exists$) The lemma provides $f \in FLT$ sending $0, 1, \infty \mapsto z_1, z_2, z_3$.

Since $\hat{\mathbb{R}} \in \mathcal{C}$, part (a) of the Theorem

$\mathbb{R} \cup \{\infty\}$

implies $f(\hat{\mathbb{R}}) \in \mathcal{C}$.

z_1, z_2, z_3

So take $C = f(\hat{\mathbb{R}})$.

(!) If there are two, then applying f^{-1} gives 2 elements of $\mathcal{C}$ through $0, 1, \infty$. But there is no circle in $\mathbb{C}$ through these points, and the only line is $\mathbb{R}$. $\square$

(from Theorem)

Proof of (b): Given $C, C' \in \mathcal{C}$, take

$$\begin{cases} z_1, z_2, z_3 \in C \text{ distinct} \\ w_1, w_2, w_3 \in C' \text{ distinct} \end{cases}.$$

Let f send $z_i \mapsto w_i$ $(\forall i)$.

Then $f(C) \in \mathcal{C}$ and contains the $\{w_i\}$.

By the Corollary, $f(C) = C'$. $\square$

Coming out of this discussion are 3 things we'd like to investigate in the next lecture:

- cross-ratio
- symmetry & orientation (if $C \mapsto C'$, what about points not on C?)
- fixed points.