Lecture 1: Complex numbers

I. Cartesian form

A <u>field</u> is a set with

- 2 commutative, associative binary operations $(+, \cdot)$
- identity elements "0" for $+$, "1" for $\cdot$
- inverses: $(-a)$ for $+$, $\underbrace{\left(\frac{1}{a}\right) \text{ for } \cdot}_{\text{if } a \neq 0}$
- distributive law.

It is <u>algebraically closed</u> if for every polynomial equation

$$a_n x^n + \dots + a_1 x + a_0 = 0$$

with $a_i \in$ the field, there exists a solution in the field.

$\mathbb{R}$ is a field, but is not algebraically closed: $x^2 + 1 = 0$.

Define an algebraically closed[†] field containing $\mathbb{R}$ by

$$\mathbb{C} := \{ a+bi \mid a,b \in \mathbb{R} \} \quad \begin{pmatrix} \text{as a set} \\ \cong \mathbb{R}^2 \end{pmatrix}$$

with

- addition: $(a_1+b_1 i) + (a_2+b_2 i) := (a_1+a_2) + (b_1+b_2)i$
  additive inverse $\quad -(a+bi) := (-a) + (-b)i$
- multiplication: $(a_1+b_1 i)\cdot(a_2+b_2 i) :=$

$$(a_1a_2 \underbrace{- b_1b_2}_{\text{treat } i \text{ as formal solution to } x^2=-1, \text{ i.e. } \sqrt{-1}}) + (a_1b_2 + a_2b_1)i$$

multiplicative inverse? $(a+bi)(a-bi) = a^2+b^2$,

so $\quad (a+bi)\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\right) = 1$.

$$\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\right) =: \frac{1}{a+bi} \text{ or } (a+bi)^{-1},$$

defined iff a,b not both 0.

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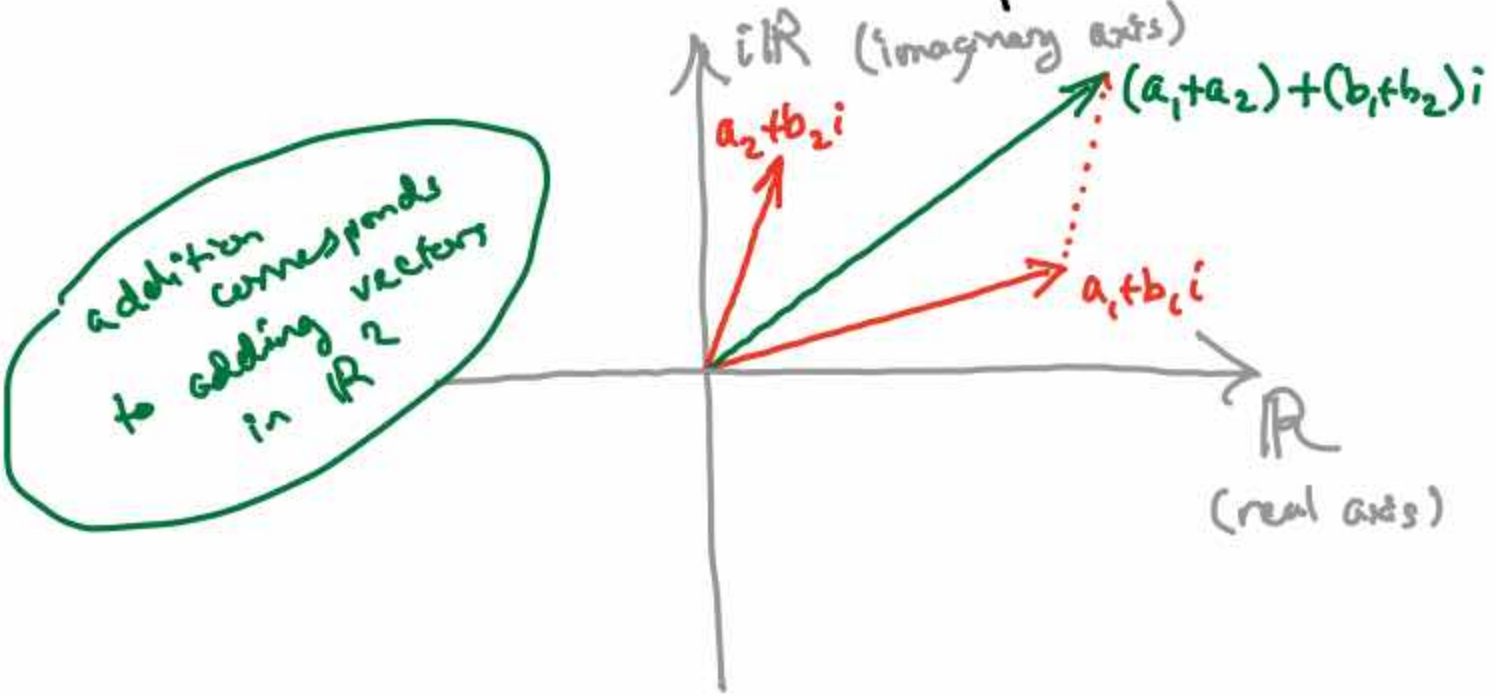
† we won't prove this now

The "embedding map" $\mathbb{R} \hookrightarrow \mathbb{C}$

is given by $a \longmapsto a + 0i$,

and respects $+, \cdot$ (i.e. is a "homomorphism of fields").

We visualize $\mathbb{R} \subseteq \mathbb{C}$ via the picture



Now define $\underline{\text{complex conjugation}}$:= flip about $\mathbb{R}$-axis.

Re, Im := $\underline{\text{projections}}$ to $\mathbb{R}$ and $i\mathbb{R}$.

$\alpha \overset{\in \mathbb{C}}{=} a+bi \implies \overline{\alpha} := a - bi$. Respects $+, \cdot$:

$\overline{\alpha_1+\alpha_2} = \overline{\alpha_1}+\overline{\alpha_2}$, $\overline{\alpha_1\alpha_2} = \overline{\alpha_1}\cdot\overline{\alpha_2}$ (HW), $\overline{\overline{\alpha}} = \alpha$.

(add then flip) (flip then add)

$\mathrm{Im}(\alpha) := b$, $\mathrm{Re}(\alpha) := a$.

$\alpha + \overline{\alpha} = (a+bi)+(a-bi) = 2a \implies$

$\alpha - \overline{\alpha} = (a+bi)-(a-bi) = (2b)i \implies$

$$\boxed{\begin{aligned} \mathrm{Re}(\alpha) &= \frac{\alpha+\overline{\alpha}}{2} \\ \mathrm{Im}(\alpha) &= \frac{\alpha-\overline{\alpha}}{2i} \end{aligned}} \quad (*)$$

The <u>modulus</u> (or <u>norm</u>, or <u>absolute value</u>) of a complex number $\alpha (= a+bi)$ is $|\alpha| := \sqrt{a^2+b^2}$, and may be thought of as the distance from 0 to α (or the length of the corresponding vector). We have

$$\alpha\bar{\alpha} = (a+bi)(a-bi) = a^2+b^2+(ab-ab)i = |\alpha|^2$$

$$\Longrightarrow \alpha \cdot \frac{\bar{\alpha}}{|\alpha|^2} = 1 \Longrightarrow \alpha^{-1} = \frac{\bar{\alpha}}{|\alpha|^2}.$$

In practice, you'll just write:

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

<u>Properties of $|\cdot|$</u>

- $|\bar{\alpha}| = |\alpha|$
- $|\alpha\beta| = |\alpha||\beta|$ $\left(|\alpha\beta|^2 = (\alpha\beta)(\overline{\alpha\beta}) = (\alpha\bar{\alpha})(\beta\bar{\beta}) = |\alpha|^2|\beta|^2 \right)$
- <u>triangle inequality</u>: $|\alpha+\beta| \le |\alpha|+|\beta|$

$$\left(\xRightarrow{\text{induction}} \left|\sum \alpha_i\right| \le \sum |\alpha_i| \; ; \; \text{also} \quad \xRightarrow[\alpha=(\alpha-\beta)+\beta]{} |\alpha-\beta| \ge \big||\alpha|-|\beta|\big| \right)$$

<u>Proof of Δ inequality</u>: for $z = x+iy \in \mathbb{C}$,

$$\mathrm{Re}(z) = x \le |x| = \sqrt{x^2} \le \sqrt{x^2+y^2} = |z|$$

Now $|\alpha+\beta|^2 = (\alpha+\beta)\overline{(\alpha+\beta)} = \alpha\bar{\alpha} + \beta\bar{\beta} + \alpha\bar{\beta} + \bar{\alpha}\beta$

$$= |\alpha|^2 + |\beta|^2 + \underbrace{2\,\mathrm{Re}(\alpha\bar{\beta})}_{\leq 2|\alpha\bar{\beta}| = 2|\alpha||\bar{\beta}| = 2|\alpha||\beta|}$$

$$\leq |\alpha|^2 + |\beta|^2 + 2|\alpha||\beta| = (|\alpha|+|\beta|)^2,$$

take $\sqrt{}$. $\square$

Cauchy-Schwarz inequality

$\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\underline{\beta}$

Define an "Hermitian inner product" on $\mathbb{C}^n$ by

$$\langle \underline{\alpha}, \underline{\beta} \rangle := \underline{\alpha} \cdot \overline{\underline{\beta}} \quad \left(= \sum_{i=1}^{n} \alpha_n \bar{\beta}_n \right)$$

and write $\|\underline{\alpha}\| := \sqrt{\langle \underline{\alpha}, \underline{\alpha} \rangle}$. This is ≥ 0, with equality iff $\underline{\alpha} = \underline{0}$. Note $\overline{\langle \underline{\alpha}, \underline{\beta} \rangle} = \langle \underline{\beta}, \underline{\alpha} \rangle$.

Let $\underline{\alpha}, \underline{\beta} \in \mathbb{C}^n$ and compute:

$$0 \leq \left\| \|\underline{\beta}\| \underline{\alpha} - \frac{\langle \underline{\alpha}, \underline{\beta} \rangle}{\|\underline{\beta}\|} \underline{\beta} \right\|^2$$

$$= \left\langle \|\underline{\beta}\| \underline{\alpha} - \frac{\langle \underline{\alpha}, \underline{\beta} \rangle}{\|\underline{\beta}\|} \underline{\beta},\ \|\underline{\beta}\| \underline{\alpha} - \frac{\langle \underline{\alpha}, \underline{\beta} \rangle}{\|\underline{\beta}\|} \underline{\beta} \right\rangle$$

$$= \|\underline{\beta}\|^2 \|\underline{\alpha}\|^2 - \langle \underline{\alpha}, \underline{\beta} \rangle \underbrace{\langle \underline{\beta}, \underline{\alpha} \rangle}_{= \overline{\langle \underline{\alpha}, \underline{\beta} \rangle}} \underbrace{- \langle \underline{\alpha}, \underline{\beta} \rangle \overline{\langle \underline{\alpha}, \underline{\beta} \rangle} + \frac{|\langle \underline{\alpha}, \underline{\beta} \rangle|^2 \cancel{\langle \underline{\beta}, \underline{\beta} \rangle}}{\cancel{\|\underline{\beta}\|^2}}}_{\text{terms cancel}}$$

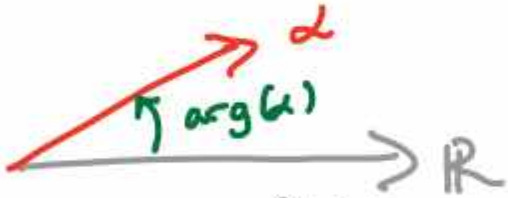
$$= \|\underline{\beta}\|^2 \|\underline{\alpha}\|^2 - |\langle \underline{\alpha}, \underline{\beta} \rangle|^2$$

$$\Longrightarrow \boxed{|\langle \underline{\alpha}, \underline{\beta} \rangle| \leq \|\underline{\beta}\| \|\underline{\alpha}\|, \text{ with equality iff } \underline{\alpha} = \gamma \underline{\beta} \text{ for some } \gamma \in \mathbb{C}}$$

Remark: If $\underline{\alpha}, \underline{\beta} \in \mathbb{R}^n$, then $\arccos\left(\frac{\langle \underline{\alpha}, \underline{\beta} \rangle}{\|\underline{\alpha}\| \|\underline{\beta}\|}\right)$ gives the angle between them.

II. Polar form

The argument $\arg(\alpha)$ of a complex number is the angle it makes with the $\mathbb{R}_{\geq 0}$-axis



which of course is $\arctan(b/a)$. We may then write

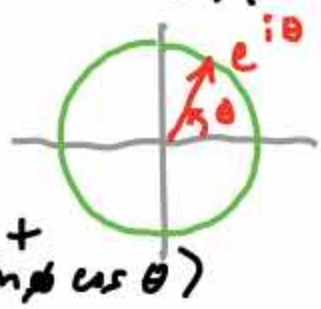
$$\alpha = |\alpha| \left(\cos(\arg(\alpha)) + i \sin(\arg(\alpha))\right)$$

← has modulus 1 itself, as $\cos^2 + \sin^2 = 1$

in polar form. Note that $\arg(\alpha)$ is defined mod 2π. (or "mod $2\pi\mathbb{Z}$")

Claim: $\arg(\alpha_1 \alpha_2) \equiv \arg(\alpha_1) + \arg(\alpha_2) \mod 2\pi$.

Proof: Define $e^{i\theta} := \cos\theta + i\sin\theta$ for $\theta \in \mathbb{R}$.



Then
$$e^{i\theta} e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$
$$= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \sin\phi\cos\theta)$$

$$= \cos(\theta+\phi) + i\sin(\theta+\phi)$$
$$= e^{i(\theta+\phi)}.$$

In particular, $(e^{i\phi})^{-1} = e^{-i\phi} \left(= \overline{e^{i\phi}}\right)$ since

$$e^{i\phi}e^{-i\phi} = e^{i(\phi-\phi)} = e^{i0} = 1.$$

So
$$\begin{aligned} e^{i\theta} = e^{i\phi} &\iff e^{i(\theta-\phi)} = 1 \\ &\iff \cos(\theta-\phi) = 1 \\ &\iff \theta \equiv \phi \mod 2\pi, \end{aligned}$$

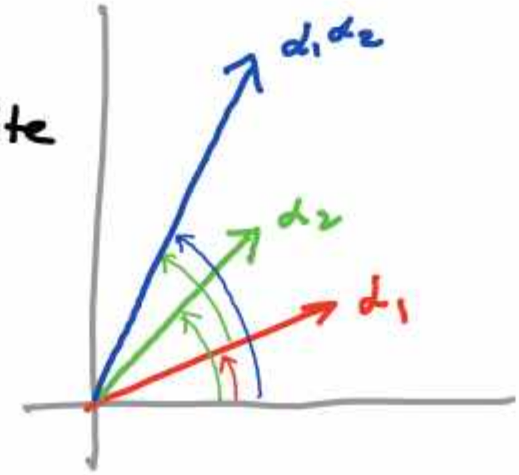
and
$$\begin{aligned} |\alpha_1\alpha_2|\, e^{i(\arg\alpha_1+\arg\alpha_2)} &= \left(|\alpha_1|e^{i\arg\alpha_1}\right)\left(|\alpha_2|e^{i\arg\alpha_2}\right) \\ &= \alpha_1\alpha_2 \\ &= |\alpha_1\alpha_2|\, e^{i\arg(\alpha_1\alpha_2)} \end{aligned}$$

$$\Rightarrow e^{i(\arg\alpha_1+\arg\alpha_2)} = e^{i\arg(\alpha_1\alpha_2)}$$

$$\Rightarrow \arg(\alpha_1)+\arg(\alpha_2) \underset{(\mathrm{mod}\ 2\pi)}{\equiv} \arg(\alpha_1\alpha_2).$$

$\square$

Hence multiplying complex numbers adds angles and multiplies lengths.



Simplifying notation, we will write

$$\alpha_j = r_j e^{i\theta_j}.$$

Quotients then look like

$$\frac{\alpha_1}{\alpha_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

Roots: Solve $z^n = \alpha$, for $\alpha = re^{i\theta}$; i.e. find $\alpha^{1/n}$.

$z = \sqrt[n]{r}\, e^{i(\theta/n)}$ is ONE solution (but see HW).

- $\underline{n=2}$: $e^{i(\theta/2)} = \cos(\theta/2) + i\sin(\theta/2) = \sqrt{\frac{1+\cos\theta}{2}} + i\sqrt{\frac{1-\cos\theta}{2}}$.

Conversions:

- polar $\to$ cartesian: $re^{i\theta} = \underbrace{r\cos\theta}_{a} + i\underbrace{r\sin\theta}_{b}$
- polar $\leftarrow$ cartesian: $a+ib = \underbrace{\sqrt{a^2+b^2}}_{r}\, e^{i\overbrace{\arctan(b/a)}^{\theta}}$

So given $\alpha = a+bi$, we find

$$\begin{aligned}
\alpha^{1/2} &= r^{1/2} e^{i\theta/2} \\
&= r^{1/2}\sqrt{\frac{1+\cos\theta}{2}} + i r^{1/2}\sqrt{\frac{1-\cos\theta}{2}} \\
&= \sqrt{\frac{r+r\cos\theta}{2}} + i\sqrt{\frac{r-r\cos\theta}{2}} \\
&= \sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}.
\end{aligned}$$

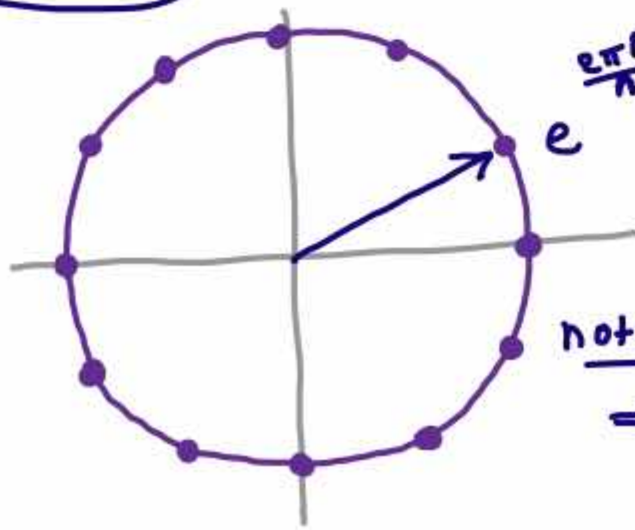
Example //

$$\sqrt{\underbrace{\overset{a}{3}+\overset{b}{4}i}_{\alpha}} = \sqrt{\frac{5+3}{2}} + i\sqrt{\frac{5-3}{2}} = 2+i.$$

(Check: $(2+i)(2+i) = (4-1) + i(2+2)$.) //

Roots of unity: Solve $z^n = 1$ ("unity"), get n^{th} roots of unity (there are n): $z = e^{i\frac{2\pi k}{n}}$, $k = 0, 1, \dots, n-1$.

$1, \zeta_n, \zeta_n^2, \dots$
$\dots, \zeta_n^{n-1}$



$e^{\frac{2\pi i}{n}} =: \zeta_n$

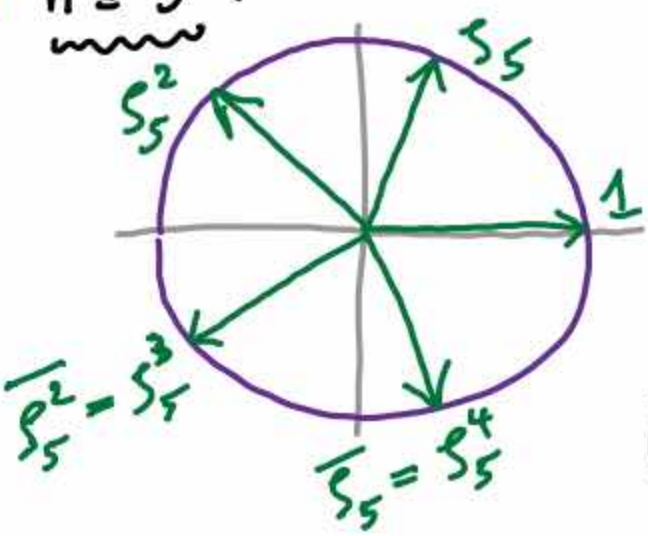
note: $\zeta_n^{n-1}\zeta_n = \zeta_n^n = 1$
$\Rightarrow \zeta_n^{n-1} = \zeta_n^{-1} = \overline{\zeta_n}$.

Examples //

- $n = 2$: $1, -1$
- $n = 3$: $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$, 1
- $n = 4$: $1, -1, i, -i$

After $n = 4$, they get harder to determine in Cartesian form:

- $n = 5$:



$\zeta_5 = a + bi$, $a^2 + b^2 = 1$

$\zeta_5^2 = (a^2 - b^2) + 2abi$
$= (2a^2 - 1) + 2abi$

$(\zeta_5^2)\zeta_5 = (\dots) + (2a^2b + (2a^2 - 1)b)i$

Now $(\zeta_5^2)\zeta_5 = \zeta_5^3 = \overline{\zeta_5^2} \Rightarrow \mathrm{Im}[(\zeta_5^2)\zeta_5] = \mathrm{Im}(\overline{\zeta_5^2})$

$$\Rightarrow 2a^2\cancel{b} + (2a^2-1)\cancel{b} = -2a\cancel{b}$$

$$\Rightarrow 4a^2 + 2a - 1 = 0$$

$$\Rightarrow \begin{cases} a = \dfrac{-2 \pm \sqrt{4+16}}{8} = \dfrac{-1 \pm \sqrt{5}}{4} \\ b = \sqrt{1-a^2} = \sqrt{1-\left(\dfrac{\sqrt{5}-1}{4}\right)^2} = \sqrt{\dfrac{5+\sqrt{5}}{8}} . \end{cases}$$

Hence, $$\zeta_5 = \frac{-1+\sqrt{5}}{4} + i\sqrt{\frac{5+\sqrt{5}}{8}}.$$ (These are cos and sin of $360/5 = 72°$.) //

Finally, recalling that

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} = \frac{1-\alpha^n}{1-\alpha},$$

we have $$1 + \zeta_n + \zeta_n^2 + \cdots + \zeta_n^{n-1} = \frac{1-\zeta_n^n}{1-\zeta_n} = \frac{0}{1-\zeta_n} = 0 .$$

III. Spherical representation (the Riemann sphere)

For many reasons — topological, geometric, analytic — it is convenient to "compactify" (or "projectively complete") $\mathbb{C}$ by adding a "point at ∞". Writing

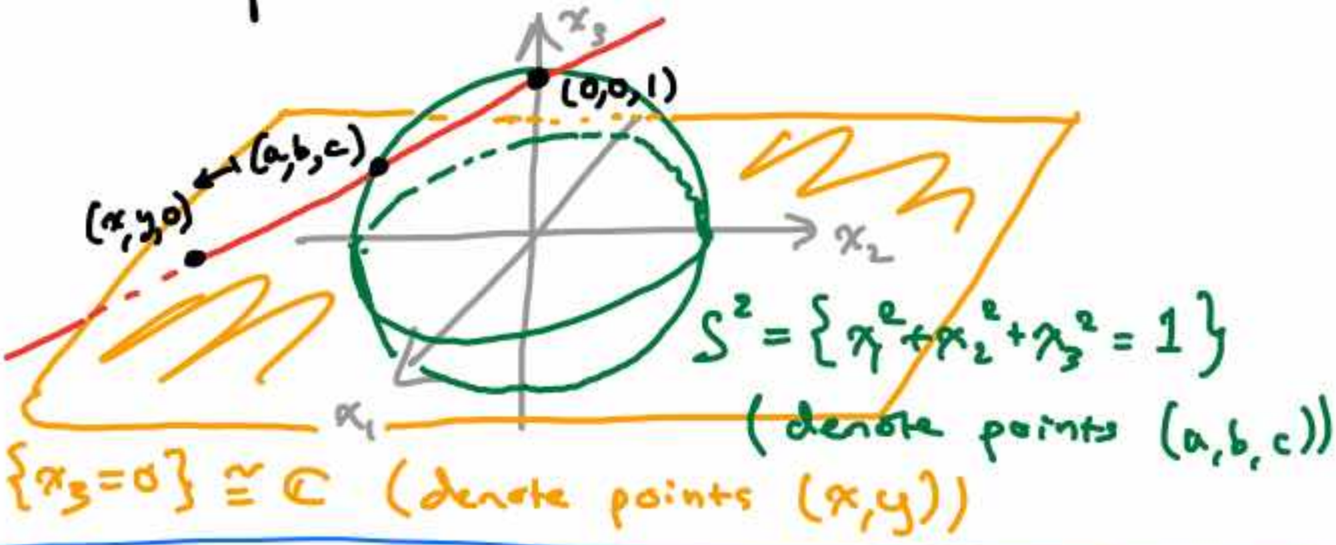
$$\hat{\mathbb{C}} := \mathbb{C} \amalg \{\infty\},$$

we topologize the result by defining basic open neighborhoods of ∞ to be sets of the form

$$\{ z \in \mathbb{C} \mid |z| > R \} \amalg \{\infty\},$$

and letting $\mathbb{C}$ have the usual Euclidean topology.†

To get a geometric model for $\hat{\mathbb{C}}$, consider the picture in $\mathbb{R}^3$



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† Another way to think of this is as the quotient

$$\frac{\mathbb{C}^2 \setminus \{\underline{0}\}}{(z_1, z_2) \sim_{\alpha \in \mathbb{C}^*} (\alpha z_1, \alpha z_2)}$$

(usually called $\mathbb{P}^1$, the complex projective line).

and define the <u>stereographic projection</u> map

$$\pi : S^2 \longrightarrow \hat{\mathbb{C}}$$

by

$$(0,0,1) \longmapsto \infty$$

$$\Big((0,0,1) \neq\Big)(a,b,c) \longmapsto L_{(a,b,c),\,(0,0,1)} \cap \{x_3 = 0\}$$

$$\|$$

$$\Big\{t \mapsto \big(at, bt, 1+(c-1)t\big)\Big\} \cap \{x_3 = 0\}$$

solution at $t = \frac{1}{1-c}$ $\longrightarrow$ $\|$

$$\left(\frac{a}{1-c}, \frac{b}{1-c}, 0\right)$$

$$\|$$

$$\left(\begin{array}{c}\text{as a point}\\ \text{of } \mathbb{C}\end{array}\right) \quad \frac{a}{1-c} + i\,\frac{b}{1-c}$$

So: $\quad \pi(a,b,c) := \dfrac{a+ib}{1-c} \quad (= z)$.

To find the inverse map, note

$$|z| = \frac{\sqrt{a^2+b^2}}{|1-c|} \underset{\substack{\uparrow\\ a^2+b^2+c^2=1}}{=} \frac{\sqrt{1-c^2}}{|1-c|} = \sqrt{\frac{1+c}{1-c}}$$

This implies

(1) that "level curves" $\{c = \text{constant}\} \subset S^2$

map to circles $\{|z| = \text{constant}\}$ centered about $0 \in \mathbb{C}$,

(2) that the upper hemisphere $(c>0)$ of S^2 maps to $|z|>1$
& the lower hemisphere $(c<0)$ of S^2 maps to $|z|<1$

(3) $|z|^2 = \dfrac{1+c}{1-c} \implies |z|^2 - |z|^2 c = 1+c$

$$\implies \frac{|z|^2-1}{|z|^2+1} = c$$

hence
$$\begin{cases} a = (1-c)\,\mathrm{Re}(z) = \dfrac{\cancel{2}}{|z|^2+1}\cdot\dfrac{z+\bar{z}}{\cancel{2}} = \dfrac{z+\bar{z}}{|z|^2+1} \\[2ex] b = (1-c)\,\mathrm{Im}(z) = \dfrac{2}{|z|^2+1}\cdot\dfrac{z-\bar{z}}{2i} = \dfrac{z-\bar{z}}{i(|z|^2+1)} \end{cases}$$

give π^{-1}, and we conclude

(4) that π is one-to-one.

(1) & (4) essentially yield part (a) of

$\boxed{\text{Theorem}}$ (a) <u>π is a homeomorphism (= 1-1, continuous map) between S^2 and $\hat{\mathbb{C}}$.</u>
<u>Either one is called the Riemann sphere.</u>

(b) $\left\{\begin{array}{l}\underline{\text{circles}} \\ \underline{\text{lines}}\end{array}\right.$ <u>in $\mathbb{C}$ correspond (under π)</u>

to circles on S^2 $\begin{cases} \text{not through } (0,0,1) \\ \text{passing through } (0,0,1) \end{cases}$.

Sketch of (b): For the lines, there are 1-to-1 correspondences

$$\begin{tikzcd}[column sep=small]
 & \text{planes through } (0,0,1) \text{ not parallel to } \{x_3=0\} \arrow[dl, "\text{intersect with } \{x_3=0\}"'] \arrow[dr, "\text{intersect with } S^2"] & \\
\text{lines in } \{x_3=0\} & & \text{circles on } S^2 \text{ passing through } (0,0,1)
\end{tikzcd}$$

For the circles, start with $C \subset S^2$. A circle lives in a plane, so must be of the form

$$C = S^2 \cap \{a_1x_1 + a_2x_2 + a_3x_3 = a_0\},$$

whose image under π is

$$\left\{ a_1 \frac{z+\bar{z}}{|z|^2+1} + a_2 \frac{z-\bar{z}}{i(|z|^2+1)} + a_3 \frac{|z|^2-1}{|z|^2+1} = a_0 \right\} \subseteq \mathbb{C}.$$

Writing $z = x+iy$, this is (after multiplying through by $|z|^2+1$)

$$2a_1 x + 2a_2 y + a_3(x^2+y^2-1) = a_0(x^2+y^2+1)$$

or

$$0 = (a_0 - a_3)(x^2+y^2) - 2a_1 x - 2a_2 y + (a_0+a_3),$$

the equation of a circle.† Conversely, given any equation of the form $0 = A(x^2+y^2) + Bx + Cy + D$, we can solve for a_0, a_1, a_2, a_3 (non-uniquely) hence realize its solution set as $\pi(\text{circle})$. $\square$

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† Recall $Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$ is the equation of a general conic. This yields a circle (or a point or the empty set — which we needn't worry about) if $C = 0$ and $A = B$.