# Lecture 50: Integration on manifolds

## Pullbacks

How do differential forms behave under smooth maps?

Given $U, V \subset \mathbb{R}^3$ open, $\vec{G} : \underset{\text{coords. } u,v,w}{V} \longrightarrow \underset{x,y,z}{U}$ $C^\infty$ and a function (0-form) $f$ on $U$,

$$\vec{G}^* f := f \circ \vec{G} \quad \text{"pullback of } f\text{"}$$

defines a function on $V$. Now writing $\vec{G}(u,v,w) = \begin{pmatrix} X(u,v,w) \\ Y(u,v,w) \\ Z(u,v,w) \end{pmatrix}$

we define

$$\vec{G}^* dx := dX = X_u\, du + X_v\, dv + X_w\, dw$$

$$\vec{G}^* dy := dY, \quad \vec{G}^* dz := dZ.$$

One then defines $\vec{G}^* \omega$ by applying $\vec{G}^*$ to every function and $dx, dy, dz$ in $\omega$. This gives an $\mathbb{R}$-linear map

$$\vec{G}^* : \Omega^k(U) \to \Omega^k(V)$$

$$\omega \longmapsto \vec{G}^* \omega$$

called the pullback.

**FACT:** Pullback and exterior derivative commute:

$$\boxed{d(\vec{G}^* \omega) = \vec{G}^* d\omega.}$$

Proof: $d(\vec{G}^* dx) = d(dX) = 0 = \vec{G}^*(d(dx))$ (& same for $dy, dz$),

so one only needs to check this on functions:

$$d(\vec{G}^* f) = d(f\circ\vec{G}) = (f\circ\vec{G})_u\, du + (f\circ\vec{G})_v\, dv + (f\circ\vec{G})_w\, dw$$

$$= (f_x X_u + f_y Y_u + f_z Z_u)\, du + (f_x X_v + f_y Y_v + f_z Z_v)\, dv + (f_x X_w + f_y Y_w + f_z Z_w)\, dw$$

$$= f_x (X_u\, du + X_v\, dv + X_w\, dw) + f_y (Y_u\, du + Y_v\, dv + Y_w\, dw) + f_z (Z_u\, du + Z_v\, dv + Z_w\, dw)$$

$$= f_x\, dX + f_y\, dY + f_z\, dZ$$

where we are thinking of $f_x$ as $f_x\circ\vec{G}$ etc.

$$= \vec{G}^*(df).$$

$\square$

Pullbacks are closely related to the Jacobian determinants that appear in the change-of-variable formulas: e.g.

$$\vec{G}^*(dx\wedge dy\wedge dz) = (X_u\, du + X_v\, dv + X_w\, dw)\wedge(Y_u\, du + Y_v\, dv + Y_w\, dw)\wedge(Z_u\, du + Z_v\, dv + Z_w\, dw)$$

$$= \begin{vmatrix} X_u & X_v & X_w \\ Y_u & Y_v & Y_w \\ Z_u & Z_v & Z_w \end{vmatrix} du\wedge dv\wedge dw.$$

## Relation to integration

Suppose $\vec{G}$ maps a bounded region $\mathcal{T} \subset V$ to $\mathcal{S} \subset U$ in 1-to-1 fashion. By the change-of-variables theorem,

$$\int_{\mathcal{S}} f \, \overset{dx\,dy\,dz}{dV} = \int_{\mathcal{T}} (f \circ \vec{G}) \, |\det J_{\vec{G}}| \, \overset{du\,dv\,dw}{dV}$$

which translates to

$$\boxed{\int_{\vec{G}(\mathcal{T})} \omega = \int_{\mathcal{T}} \vec{G}^{*} \omega \, .}$$

This works in any dimension, and allows us to <u>define</u> integrals on manifolds.

<u>Definition</u>: Let $M \subset \mathbb{R}^m$ be a closed, bounded set. Suppose that every point $p \in M$ has an open neighborhood $B \subset \mathbb{R}^m$ and a 1-1 $C^\infty$ map $\vec{g} : \mathcal{U} \to \mathbb{R}^m$ with image $B \cap M$, where $\mathcal{U} \subset \mathbb{R}^n$ is either an open ball centered at $\vec{0}$ or its intersection with $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$, and $\operatorname{rank}(J_{\vec{g}}) = n$ on all of $\mathcal{U}$. Then $M$ is called a <u>compact $n$-manifold</u>.

For simplicity, assume that we have a single parametrizing map $\vec{g} : \mathcal{T} \to \mathbb{R}^n$ with image $\vec{g}(\mathcal{T}) = M$.

smooth, 1-1

$\mathcal{T} \subset \mathbb{R}^n$

Let $\omega \in \Omega^n(U)$ be an $n$-form on a set $U \subset \mathbb{R}^n$ containing $M$. Then we define

$$(*) \qquad \boxed{\int_M \omega := \int_{\mathcal{J}} \vec{g}^* \omega.}$$

(One can check this is independent of the choice of parametrization.)

This is exactly how we defined line and surface integrals (and again, generalizes to any dimension). For instance, if $\vec{r} : [a,b] \to \mathbb{R}^2$ parametrizes a curve $C$, then $(*)$ says

$t \mapsto \vec{r}(t) = (x(t), y(t))$

$$\int_C \underbrace{f\,dx + g\,dy}_{\omega} := \int_a^b \vec{r}^* \omega = \int_a^b f(\vec{r}(t))\,dx(t) + g(\vec{r}(t))\,dy(t)$$

$$= \int_a^b f(\vec{r}(t))\,x'(t)\,dt + f(\vec{r}(t))\,y'(t)\,dt.$$

Finally, we can state the most general form of

Stokes's Theorem: If $M$ is a compact $n$-manifold† with boundary $\partial M$ (= $(n-1)$ manifold), and $\omega$ is an $(n-1)$-form on $M$, then

$$\int_M d\omega = \int_{\partial M} \omega.$$

† Technically one should also choose an orientation of $M$ so that the sign of the integral is well-defined.

Ex/ Returning to the example at the end of lecture 49,

$$\omega = \frac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}} \in \Omega^2(\mathbb{R}^3\setminus\{\vec{0}\})$$

has $d\omega = 0$. Its restriction to a sphere $S_r$ of radius $r$ about $\vec{0}$ has $\int_{S_r}\omega = 4\pi$, as we showed before. So $\int_{S_r} r^2\omega = 4\pi r^2 = a(S_r)$ and we call $r^2\omega$ $\left(= \frac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{\sqrt{x^2+y^2+z^2}}\right)$ the <u>area form</u> on (any) $S_r$.

If $\mathcal{S} \subset \mathbb{R}^3$ is a closed surface $(\partial[\mathcal{S}] = \emptyset)$ enclosing $\vec{0}$, then for $r=\epsilon$ sufficiently small it encloses $S_\epsilon$ too, and there is a solid $M$ with $\partial[M] = \mathcal{S} - S_\epsilon$. Since $d\omega = 0$, Stokes gives


$$0 = \int_M d\omega = \int_{\partial[M]}\omega = \int_{\mathcal{S}}\omega - \int_{S_\epsilon}\omega$$

$$= \int_{\mathcal{S}}\omega - 4\pi$$

$$\Rightarrow \int_{\mathcal{S}}\omega = 4\pi.$$

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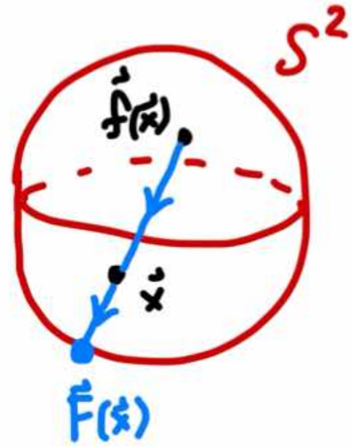
# A fixed-point theorem




Let $D^3 := \{\vec{x} \in \mathbb{R}^3 \mid \|\vec{x}\| \le 1\}$

$S^2 := \{\vec{x} \in \mathbb{R}^3 \mid \|\vec{x}\| = 1\}$

so that $\partial D^3 = S^2$.

We have the area form (note $r=1$, so there's no $\sqrt{x^2+y^2+z^2}$ in the denominator)

$$\sigma = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy \in \Omega^2(S^2),$$

whose integral $\displaystyle\int_{S^2} \sigma \overset{(\dagger)}{=} 4\pi$.

<u>Theorem</u>: If $\vec{f} : D^3 \to D^3$ is $C^\infty$, then it has a fixed point: i.e., $\vec{f}(\vec{a}) = \vec{a}$ for some $\vec{a} \in D^3$.

<u>Proof</u>: Suppose $\vec{f}$ has no fixed point. Then we can define a $C^\infty$ map $\vec{F} : D^3 \to S^2$ as shown in the picture. Clearly if $\vec{x} \in S^2$, then $\vec{F}(\vec{x}) = \vec{x}$; so $\vec{F}\big|_{S^2} = id_{S^2}$. ← identity map

Hence
$$\int_{S^2} \sigma = \int_{\vec{F}(S^2)} \sigma = \int_{S^2} F^*\sigma = \int_{\partial D^3} F^*\sigma$$
$$\underset{\text{Stokes}}{=} \int_{D^3} d(F^*\sigma) = \int_{D^3} F^*(\underbrace{d\sigma})$$
$$= 0.$$

As a form on $S^2$ this is 0, because $\Omega^3(S^2) = \{0\}$ (think: pullback to $U \subset \mathbb{R}^2$ under a parametrization)

But this contradicts $(\dagger)$. $\square$