Lecture 49 : Differential Forms

Today I want to tell you how a change in perspective unifies Gauss, Stokes, and the Fundamental Theorem of Calculus for line integrals. I've already suggested why they are all analogous; now we'll introduce the "Calculus" which establishes their unity and makes higher-dimensional generalizations immediate.

Vector Spaces (real, of course)

$\Omega^0 := \mathbb{R}$ = 1-dim'l vector space with basis $\{1\}$

$\Omega^1 :=$ 3-dim'l vector space with basis $\{dx, dy, dz\}$

$\Omega^2 :=$ 3-dim'l vector space with basis $\{dy\wedge dz, dz\wedge dx, dx\wedge dy\}$

these are just names of vectors, nothing more

write $dy \wedge dx = -dx\wedge dy$ etc.

$\Omega^3 :=$ 1-dim'l vector space with basis $\{dx\wedge dy\wedge dz\}$

$= -dy\wedge dx\wedge dz = dy\wedge dz\wedge dx = \ldots$

Ω^4 & higher are $\{0\}$.

Differential forms $U\subset\mathbb{R}^3$ open set

A (smooth) differential k-form is a (C^∞, vector-valued) function from U to Ω^k. We write

$$\omega\in\Omega^k(U).$$

$\underline{k=0}$: A 0-form is a C^∞ function $f(x,y,z)$. That is, $\Omega^0(U) = C^\infty(U)$.

$\underline{k=1}$: A 1-form $\omega \in \Omega^1(U)$ takes the form

$$\omega = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.$$

$\underline{k=2}$: A 2-form $\omega \in \Omega^2(U)$ takes the form

$$\omega = P(x,y,z)\,dy\wedge dz + Q(x,y,z)\,dz\wedge dx + R(x,y,z)\,dx\wedge dy.$$

$\underline{k=3}$: A 3-form $\omega \in \Omega^3(U)$ takes the form

$$\omega = f(x,y,z)\,dx\wedge dy\wedge dz.$$

$\boxed{\text{N.B. } k \text{ is called the degree of a differential } (k\text{-}) \text{ form.}}$

Exterior derivative

If $f \in \Omega^0(U)$ is a function, define a 1-form $df \in \Omega^1(U)$ by

$$df := f_x\,dx + f_y\,dy + f_z\,dz.$$

We can extend this to higher degree, obtaining (for each k) linear maps

$$d : \Omega^k(U) \to \Omega^{k+1}(U).$$

Here's how: require that d be zero on the basis vectors

$$1,\ dx,\ dy,\ dz,\ dy\wedge dz,\ dz\wedge dx,\ dx\wedge dy,\ dx\wedge dy\wedge dz$$

and that d be $\mathbb{R}$-linear

i.e. $d(a\omega_1 + b\omega_2) = a\,d\omega_1 + b\,d\omega_2$ etc.

and satisfy the product rule

$$d(f\alpha) = df\wedge\alpha + f\,d\alpha.$$

So for example

$$\begin{aligned} d\underbrace{(f\,dx)}_{1\text{-form}} &= df\wedge dx + f\,\cancelto{0}{d(dx)} \\ &= (f_x\,dx + f_y\,dy + f_z\,dz)\wedge dx \end{aligned}$$

$$= f_x \underbrace{dx\wedge dx} + f_y\, dy\wedge dx + f_z\, dz\wedge dx$$

$= -dx\wedge dx$ by "swapping dx's"

$\Longrightarrow dx\wedge dx = 0$

$$= \underbrace{f_z\, dz\wedge dx - f_y\, dx\wedge dy}_{2\text{-form}},$$

and

$$d(\underbrace{f\, dx\wedge dy}_{2\text{-form}}) = df\wedge dx\wedge dy + f\, \cancel{d(dx\wedge dy)} \to 0$$

$$= f_x \overbrace{dx\wedge dx}^{=0}\wedge dy + f_y \overbrace{dy\wedge dx\wedge dy}^{0} + f_z\, dz\wedge dx\wedge dy$$

$$= \underbrace{f_z\, dx\wedge dy\wedge dz}_{3\text{-form}}.$$

(Clearly $d(f\, dx\wedge dy\wedge dz) = 0$.)

Now notice that

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U)$$

$$f \longmapsto df = f_x\, dx + f_y\, dy + f_z\, dz$$

"matches" the gradient $\vec{\nabla} f = (f_x, f_y, f_z)$,

$$\Omega^1(U) \xrightarrow{d} \Omega^2(U)$$

$$P\,dx + Q\,dy + R\,dz \longmapsto (P_y \overbrace{dy\wedge dx}^{-dx\wedge dy} + P_z\, dz\wedge dx)$$

$$+ (Q_x\, dx\wedge dy + Q_z \overbrace{dz\wedge dy}^{-dy\wedge dz})$$

$$+ (R_x \overbrace{dx\wedge dz}^{-dz\wedge dx} + R_y\, dy\wedge dz)$$

$$= (R_y - Q_z)\, dy\wedge dz + (P_z - R_x)\, dz\wedge dx + (Q_x - P_y)\, dx\wedge dy$$

"matches" $\operatorname{curl}(P,Q,R) = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y)$, and

$$\Omega^2(U) \xrightarrow{d} \Omega^3(U)$$

$$p\,dy\wedge dz + q\,dz\wedge dx + r\,dx\wedge dy \longmapsto p_x\,dx\wedge dy\wedge dz + q_y\,\underbrace{dy\wedge dz\wedge dx}_{=dx\wedge dy\wedge dz} + r_z\,\underbrace{dz\wedge dx\wedge dy}_{=dx\wedge dy\wedge dz}$$

$$= (p_x + q_y + r_z)\,dx\wedge dy\wedge dz$$

"matches" $\mathrm{div}(p,q,r) = p_x + q_y + r_z$.

So the identity $\mathrm{curl}(\vec{\nabla} f) = 0$ becomes

$$d(df) = 0,$$

and the identity $\mathrm{div}(\mathrm{curl}(\vec{F})) = 0$ becomes

$$d(d\omega) = 0. \qquad (\omega \in \Omega^1(U))$$

In all degrees we therefore have $d \circ d = 0$.

The "generalized Stokes theorem"

We already know how to integrate differential forms:

- $\underset{f}{\overset{\shortparallel}{\omega}} \in \Omega^0(U)$, $\Gamma = \underset{\text{a point in } U}{p}$ $\Longrightarrow \int_\Gamma \omega := f(p)$.

  $\Gamma = \underbrace{p - q}_{\text{a formal difference of points}}$ $\Longrightarrow \int_\Gamma \omega := f(p) - f(q)$.

- $\underset{P\,dx + Q\,dy + R\,dz}{\overset{\shortparallel}{\omega}} \in \Omega^1(U)$, $\Gamma = \underset{\text{oriented curve}}{\mathcal{C}}$ $\Longrightarrow \int_\Gamma \omega := \int_{\mathcal{C}} P\,dx + Q\,dy + R\,dz$

- $\underset{P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy}{\overset{\shortparallel}{\omega}} \in \Omega^2(U)$, $\Gamma = \underset{\text{oriented surface}}{\mathcal{S}}$ $\longrightarrow \int_\Gamma \omega := \iint_{\mathcal{S}} P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy$

• $\underset{= f\,dx\,dy\,dz}{\omega} \in \Omega^3(U)$, $\Gamma = \underset{\uparrow \text{ solid region}}{\mathcal{V}}$ $\Longrightarrow$ $\int_\Gamma \omega := \iiint_{\mathcal{V}} f\,dx\,dy\,dz$

Write "∂" for the boundary operator, viz.

Γ: 

$\partial\Gamma := \partial[\mathcal{C}] = p - q$ $\qquad$ $\partial\Gamma := \partial[\mathcal{S}] = \mathcal{C}$ $\qquad$ $\partial\Gamma := \partial[\mathcal{V}] = \mathcal{S}$

Theorem: Let Γ be a $(k+1)$-dimensional "object"†, with $\partial\Gamma$ as its k-dim'l boundary; & $\omega \in \Omega^k(U)$ a k-form. Then

$$\int_\Gamma d\omega = \int_{\partial\Gamma} \omega .$$

Proof:

$\boxed{k=0}$

$$\int_\Gamma \partial\omega \qquad\qquad \int_{\partial\Gamma} \underset{= f}{\omega}$$

$$\int_\Gamma \partial\omega = \int_{\mathcal{C}} f_x\,dx + f_y\,dy + f_z\,dz = \int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{r}$$

$$\int_{\partial\Gamma} \omega = f(p) - f(q)$$

$\int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{r} = f(p) - f(q)$: FTC for line integrals

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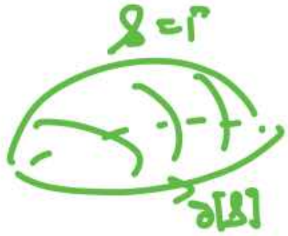
† [piecewise] smooth curve, surface, or solid, with [piecewise] smooth boundary (the technical term is "manifold with boundary")

$\boxed{k=1}$



$$\int_\Gamma d\omega \qquad\qquad \int_{\partial\Gamma} \omega \;\overset{\text{"}}{=}\; P\,dx + Q\,dy + R\,dz$$

$$\int_\Gamma d\omega = \iint_S (R_y - Q_z)\,dy\wedge dz + (P_z - R_x)\,dz\wedge dx + (Q_x - P_y)\,dx\wedge dy$$

$$= \iint_S \operatorname{curl}(P,Q,R)\cdot\hat{n}\,dS$$

$$\int_{\partial\Gamma}\omega = \int_{\partial[S]} (P,Q,R)\cdot d\vec{r}$$

$$\iint_S \operatorname{curl}(P,Q,R)\cdot\hat{n}\,dS = \int_{\partial[S]} (P,Q,R)\cdot d\vec{r}$$

Stokes's Theorem

$\boxed{k=2}$

$$\int_\Gamma d\omega \qquad\qquad \int_{\partial\Gamma} \omega \;\overset{\text{"}}{=}\; p\,dy\wedge dz + q\,dz\wedge dx + r\,dx\wedge dy$$

$$\int_\Gamma d\omega = \iiint_V (p_x + q_y + r_z)\,dx\,dy\,dz = \iiint_V \operatorname{div}(p,q,r)\,dV$$

$$\int_{\partial\Gamma}\omega = \iint_{\partial[V]} (p,q,r)\cdot\hat{n}\,dS$$

$$\iiint_V \operatorname{div}(p,q,r)\,dV = \iint_{\partial[V]} (p,q,r)\cdot\hat{n}\,dS$$

Gauss's Theorem

$\square$

"Holeyness"?

Now consider the sequence of d's

$$(*) \qquad \underbrace{\mathbb{R} \xrightarrow{\text{"}d_{-1}\text{"}} \Omega^0(U)}_{\text{this is just the inclusion of constant functions: } c \mapsto c.} \xrightarrow{d_0} \Omega^1(U) \xrightarrow{d_1} \Omega^2(U) \xrightarrow{d_2} \Omega^3(U) \xrightarrow{d_3} 0$$

and recall that $d\circ d = 0$.

- If U is connected (0-connected), then $\vec{\nabla} f = \vec{0} \Rightarrow f$ is constant.

  That is: for $\omega \in \Omega^0(U)$, $d\omega = 0 \Rightarrow \omega$ constant.

- If U is simply connected ("1-connected"), then $\mathrm{curl}(\vec{F}) = 0$ $\Longrightarrow$ $\vec{F} = \vec{\nabla} f$ for some f. Here that translates to: for $\omega \in \Omega^1(U)$, $d\omega = 0 \Longrightarrow \omega \in d(\Omega^0(U))$.
- If U is 2-connected, $\mathrm{div}(\vec{F}) = 0 \Rightarrow \vec{F} = \mathrm{curl}\, \vec{G}$ for some $\vec{G}$. Here this becomes: for $\omega \in \Omega^2(U)$,
$$d\omega = 0 \Longrightarrow \omega \in d(\Omega^1(U)).$$
- Finally, one can show that any function f is $\mathrm{div}(\vec{F})$ for some $\vec{F}$. Here this means that d_2 is surjective.

The upshot is that if U is convex (hence 0, 1, & 2-connected), then $\ker(d_k) = \mathrm{image}(d_{k-1})$ for $k = 0, 1, 2,$ & 3. This is called the "Poincaré Lemma", and makes (*) into what is called an <u>exact sequence</u>.

Conversely, the failure of "exactness of (*)" can be used as a measure of the "holeyness" of U:

Dimension of ... measures

$H^2(U) := \dfrac{\ker(d_2)}{\mathrm{im}(d_1)}$ # of "solid holes" in U

$H^1(U) := \dfrac{\ker(d_1)}{\mathrm{im}(d_0)}$ # of "tunnels" through U

$H^0(U) := \ker(d_0)$ # of connected components of U

cohomology groups of U (if U convex, first 2 are zero & $H^0(U) = \mathbb{R}$)

Corollary: If $\omega \in \Omega^k(U)$ has $d\omega = 0$, γ is a closed k-dim'l object ($\partial\gamma = 0$), and $\int_\gamma \omega \neq 0$, then $H^k(U) \neq \{0\}$.

Proof: If $H^k(U) = 0$, then $d\omega = 0 \implies \omega = d\eta$ for some $\eta \in \Omega^{k-1}(U)$. The Theorem then implies $\int_\gamma \omega = \int_\gamma d\eta = \int_{\underbrace{\partial\gamma}_{=0}} \eta = 0$, a contradiction. $\square$

Ex/ $k = 2$, $U = \mathbb{R}^3 \setminus \{(0,0,0)\}$, γ = sphere, and

$$\omega = \frac{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}.$$

We showed (in effect): $\int_\gamma \omega = 4\pi$. So $H^2(U) \neq \{0\}$. //

Everything we have done here extends to higher dimension and to spaces ("manifolds") other than open subsets of $\mathbb{R}^n$.

In the last couple of classes I will give some applications.