Monads on projective spaces

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1 Introduction

Let $\mathbf{P}^n$ denote n-dimensional projective space. If $\mathcal{E}$ is a vector bundle on $\mathbf{P}^n$ then we have an associated locally free sheaf, $\mathcal{O}_{\mathbf{P}^n}(\mathcal{E})$, of germs of sections in $\mathcal{E}$. In what follows, we will not make any distinction between a vector bundle and its associated locally free sheaf. We will say that a vector bundle, $\mathcal{E}$, *splits* if it completely decomposes as a sum of line bundles. For any sheaf, $\mathcal{E}$, on $\mathbf{P}^n$ we will let $H^i_*(\mathcal{E}) = \bigoplus_{\nu \in \mathbb{Z}} H^i(\mathbf{P}^n, \mathcal{E}(\nu))$. If $\mathcal{E}$ is a vector bundle, then the cohomology modules, $H^i_*(\mathcal{E})$, are of finite length whenever $1 \leq i \leq n-1$. There is a strong relation between $\mathcal{E}$ and these "intermediate cohomology modules". In this paper, we will be focusing on a few aspects of this relation. To begin with, the vanishing of various subsets of these modules force interesting conditions on $\mathcal{E}$. The strongest set of conditions in this direction are found in a theorem of Horrocks [Hor1] which states:

If $\mathcal{E}$ is a vector bundle on $\mathbf{P}^n$ then $\mathcal{E}$ splits if and only if $H^i_*(\mathcal{E}) = 0$ for $1 \leq i \leq n-1$.

Evans and Griffith [EG] established the following improvement of Horrocks' theorem:

If $\mathcal{E}$ is a rank k vector bundle on $\mathbf{P}^n$ with $k \leq n$ then $\mathcal{E}$ splits if and only if $H^i_*(\mathcal{E}) = 0$ for $1 \leq i \leq k-1$.

Imposing other restrictions on the cohomology modules may also force the bundle to split. For instance, in a paper by Kumar and Rao [KR], they established:

If $\mathcal{E}$ is a rank 2 bundle on $\mathbf{P}^n$ with $n \geq 4$ (resp. $n \geq 5$) then $\mathcal{E}$ splits if and only if $H^1_*(\mathcal{E})$ is Buchsbaum (resp. 2-Buchsbaum).

To the best of our knowledge, even for rank two bundles on projective spaces, the consequences of restrictions on cohomology modules other than the first have not been explored. In this note, our first result is the following cousin of the theorem of Horrocks and of the theorem of Evans and Griffith:

Theorem 1. *Let $\mathcal{E}$ be a vector bundle on $\mathbf{P}^n$.*

1. *If n is even and if $\operatorname{rank}(\mathcal{E}) < n$ then $\mathcal{E}$ splits if and only if $H^i_*(\mathcal{E}) = 0$ for $1 < i < n-1$.*
2. *If n is odd and if $\operatorname{rank}(\mathcal{E}) < n-1$ then $\mathcal{E}$ splits if and only if $H^i_*(\mathcal{E}) = 0$ for $1 < i < n-1$.*

An immediate corollary of this theorem is:

Corollary 2. *If $\mathcal{E}$ is a vector bundle on $\mathbf{P}^4$ and if* $\text{rank}(\mathcal{E}) \leq 3$ *then $\mathcal{E}$ splits if and only if* $H^2_*(\mathcal{E}) = 0$.

Our final result, along the lines of [KR], is the following:

Proposition 3. *If $\mathcal{E}$ is a non-split rank two bundle on $\mathbf{P}^4$ and if $H^2_*(\mathcal{E})$ is Buchsbaum, then $\mathcal{E}$ is a Horrocks-Mumford bundle.*

2 Main Results

Atiyah, Hitchen, Drinfeld and Manin introduced the notion of a mathematical instanton bundle on $\mathbf{P}^3$ [AHDM]. Okonek and Spindler generalized this idea to give a definition of mathematical instanton bundles on $\mathbf{P}^{2n+1}$ [OS]. In addition, they establish the existence of these bundles for different Chern classes and study their moduli space. Instanton bundles on $\mathbf{P}^{2n+1}$ were further studied by Spindler and Trautmann [ST] and by Ancona and Ottaviani [AO]. One criterion that these bundles satisfy is that they have rank $2n$ on $\mathbf{P}^{2n+1}$ and have zero i^{th} cohomology modules for $1 < i < 2n$. Hence they appear as the homology in the middle of a "monad of sums of line bundles". To be precise, each such mathematical instanton appears as the homology of a complex

$$\mathcal{O}_{\mathbf{P}}(-1)^k \to \mathcal{O}_{\mathbf{P}}{}^{2n+2k} \to \mathcal{O}_{\mathbf{P}}(1)^k,$$

where the map on the left is an inclusion of vector bundles and the map on the right is a surjection of vector bundles. Since the maps are given by matrices of linear forms, we may call such a monad a "linear monad of bundles".

One can create monads which are not "linear" by, for example, pulling back a linear monad by a finite map from $\mathbf{P}^{2n+1}$ to itself. In fact, Horrocks [Hor2] shows that every rank two bundle on $\mathbf{P}^3$ is obtained as the homology of a monad of the form

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are sums of line bundles. Additionally, the monad can be chosen to be minimal in the sense that it is built from a minimal resolution of the first cohomology module ([R2] and [D]). An analogous question can be asked for rank two bundles on $\mathbf{P}^4$. We address some cases of this question (originally asked by Decker). As a consequence of our investigation, we are able to identify all rank two bundles on $\mathbf{P}^4$ which have Buchsbaum second cohomology module.

Recently, Floystad [F] analyzed linear monads on $\mathbf{P}^n$ of the form

$$\mathcal{O}_{\mathbf{P}}(-1)^a \to \mathcal{O}_{\mathbf{P}}{}^b \to \mathcal{O}_{\mathbf{P}}(1)^c.$$

While he was interested in linear monads for sheaves as well as for bundles, (i.e. he allowed the left hand map to be an inclusion of sheaves and not necessarily an inclusion of bundles), his results for linear monads for bundles show that under precise conditions on a, b, c and n, the homology of the monad gives rise to vector bundles on $\mathbf{P}^n$. He proves that if one obtains a vector bundle, $\mathcal{E}$, on $\mathbf{P}^{2n}$ then $\text{rank}(\mathcal{E}) \geq 2n$ and if one obtains a vector bundle, $\mathcal{E}$, on $\mathbf{P}^{2n+1}$

then $\text{rank}(\mathcal{E}) \geq 2n$. Furthermore, he identifies the ones of rank $2n$ on $\mathbf{P}^{2n+1}$ with instanton bundles.

The main purpose of our note is to show that these rank limitations are true even with arbitrary monads and not just linear monads.

In the following paragraphs, when we deal with the projective space $\mathbf{P}^n$, S will denote the underlying polynomial ring, and for any sheaf $\mathcal{F}$, $H^i_*(\mathcal{F})$ will denote the S-module $\bigoplus_{\nu\in\mathbb{Z}} H^i(\mathbf{P}^n, \mathcal{F}(\nu))$.

Theorem 4. *Let $\mathcal{E}$ be a vector bundle on $\mathbf{P}^n$.*

1. *If n is even and if $\text{rank}(\mathcal{E}) < n$ then $\mathcal{E}$ splits if and only if $H^i_*(\mathcal{E}) = 0$ for $1 < i < n-1$.*
2. *If n is odd and if $\text{rank}(\mathcal{E}) < n-1$ then $\mathcal{E}$ splits if and only if $H^i_*(\mathcal{E}) = 0$ for $1 < i < n-1$.*

Proof. When $\mathcal{E}$ is a vector bundle on $\mathbf{P}^n$, it is well known that the following two conditions are equivalent:

1. $H^i_*(\mathcal{E}) = 0$ for $1 < i < n-1$.
2. There exist sums of line bundles $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ on $\mathbf{P}^n$ and a monad $\mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C}$, where α is an injection and β a surjection of vector bundles, such that $\mathcal{E}$ is given by $ker(\beta)/image(\alpha)$.

This equivalence follows from Horrocks' theorem and his method of killing $H^1_*(\mathcal{E})$ and $H^{n-1}_*(\mathcal{E})$ ([Hor2]) to produce a monad.

Suppose $\mathcal{E}$ is non-split and satisfies the two conditions. Then we can assume in condition 1 that either $H^{n-1}_*(\mathcal{E})$ or $H^1_*(\mathcal{E})$ is nonzero and in condition 2 that both α and β are minimal in the sense that no matrix entry is a non-zero scalar and at least one of these matrices is non-zero.

Now if one of $\mathcal{A}$, $\mathcal{C}$ is zero, then $\mathcal{E}$ or its dual is a first syzygy module. In this case, $\mathcal{E}$ must have rank at least n by the following well-known argument. Assume that $\mathcal{C}$ is zero and let r be the rank of $\mathcal{E}$. From the short exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{E} \to 0,$$

we get the exact sequence

$$0 \to S^r\mathcal{A} \to S^{r-1}\mathcal{A} \otimes \wedge^1\mathcal{B} \to \cdots \to \wedge^r\mathcal{B} \to \wedge^r\mathcal{E} \to 0.$$

Chasing exact sequences tells us that when $r < n$, this long exact sequence is split at each place. In particular, the map $S^r\mathcal{A} \to S^{r-1}\mathcal{A} \otimes \wedge^1\mathcal{B}$ which is obtained from α as $a_1a_2\ldots a_r \to \sum(\pm a_1a_2..\hat{a}_i..a_r \otimes \alpha(a_i))$, is split. This goes against our assumption that the matrix α had no non-zero scalars.

Suppose we know the result of the theorem for n even. Let $\mathcal{E}$ be a bundle on $\mathbf{P}^n$ with $\text{rank}(\mathcal{E}) < n-1$, $n \geq 3$, n odd. If $\mathcal{E}$ satisfies condition 1 and condition 2 and if we restrict $\mathcal{E}$ to a hyperplane H (of even dimension), then these two conditions are inherited by the restriction. By our assumption of the result of the theorem for n even, the restricted bundle is split. Since the restriction of the bundle is split, this forces the bundle on the larger space to be split ([OSS] Theorem 2.3.2). Thus, establishing the result of the theorem for the case of n even will also establish the result for n odd.

Suppose now that n is even with $n = 2k$. Let $\mathcal{E}$ be a bundle on $\mathbf{P}^n$ with $\mathrm{rank}(\mathcal{E}) \leq n-1$. By adding line bundles to $\mathcal{E}$ (if necessary), we may suppose that $\mathrm{rank}(\mathcal{E}) = n-1$. Let $c \in \mathbb{Z}$ be the first Chern class of $\mathcal{E}$.

We will study $\wedge^{k-1}\mathcal{E}$ and $\wedge^k\mathcal{E}$. Let $\mathcal{G}$ equal the kernel of the bundle surjection β in the monad. We get

$$0 \to \mathcal{G} \to \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0,$$
$$0 \to \mathcal{A} \to \mathcal{G} \to \mathcal{E} \to 0.$$

We claim that $H^k_*(\wedge^{k-1}\mathcal{E}) = 0$. To see this, first consider the short exact sequence

$$0 \to \mathcal{C}^\vee \to \mathcal{B}^\vee \to \mathcal{G}^\vee \to 0.$$

From this short exact sequence, $\wedge^i\mathcal{G}^\vee$ has a resolution by sums of line bundles:

$$0 \to S^i(\mathcal{C}^\vee) \to S^{i-1}(\mathcal{C}^\vee) \otimes \wedge^1(\mathcal{B}^\vee) \to \cdots \to \wedge^i(\mathcal{B}^\vee) \to \wedge^i(\mathcal{G}^\vee) \to 0.$$

Therefore

$$H^j_*(\wedge^i\mathcal{G}^\vee) = 0 \text{ for } 1 \leq j \leq 2k-i-1.$$

Hence

$$H^p_*(\wedge^i\mathcal{G}) = 0 \text{ for } i+1 \leq p \leq 2k-1.$$

Now from the short exact sequence

$$0 \to \mathcal{A} \to \mathcal{G} \to \mathcal{E} \to 0,$$

we have the exact sequence

$$0 \to S^{k-1}\mathcal{A} \to S^{k-2}\mathcal{A} \otimes \wedge^1\mathcal{G} \to \cdots \to \wedge^{k-1}\mathcal{G} \to \wedge^{k-1}\mathcal{E} \to 0.$$

If we break this up into short exact sequences and use the vanishing of cohomology for $\wedge^i(\mathcal{G}^\vee)$ then we obtain $H^k_*(\wedge^{k-1}\mathcal{E}) = 0$.

By Serre duality, $\wedge^{k-1}\mathcal{E}$ and $\wedge^k\mathcal{E}$ are dual to each other up to a twist by $\mathcal{O}_{\mathbf{P}}(c)$ (since $\mathrm{rank}(\mathcal{E}) = 2k-1$), hence we get

$$H^k_*(\wedge^k\mathcal{E}) = 0.$$

Consider the resolution for $\wedge^k\mathcal{E}$ obtained as

$$0 \to S^k\mathcal{A} \to S^{k-1}\mathcal{A} \otimes \wedge^1\mathcal{G} \to \cdots \to \wedge^k\mathcal{G} \to \wedge^k\mathcal{E} \to 0.$$

By our calculations, we conclude that the map $H^{2k}(S^k\mathcal{A}) \to H^{2k}(S^{k-1}\mathcal{A} \otimes \wedge^1\mathcal{G})$ is an inclusion, hence its Serre dual

$$H^0_*(S^{k-1}\mathcal{A}^\vee \otimes \wedge^1\mathcal{G}^\vee) \to H^0_*(S^k\mathcal{A}^\vee)$$

is surjective. This tells us that

$$0 \to S^k\mathcal{A} \to S^{k-1}\mathcal{A} \otimes \wedge^1\mathcal{G}$$

is a split inclusion. Hence the composite

$$0 \to S^k\mathcal{A} \to S^{k-1}\mathcal{A} \otimes \mathcal{B}$$

is also a split inclusion. This is a contradiction as we saw in the earlier part of the proof. □

As a particular case, we get

Corollary 5. *If $\mathcal{E}$ is a rank two bundle on $\mathbf{P}^4$, then $\mathcal{E}$ splits if and only if $H^2_*(\mathcal{E}) = 0$.*

This last corollary can be viewed as the simplest case of the following question due to W.Decker.

Question 6. Let $\mathcal{E}$ be a non-split rank two bundle on $\mathbf{P}^4$, and let $\mathcal{P}$ be the second syzygy bundle for the module $H^2_*(\mathcal{E})$. It follows by the killing of $H^1_*(\mathcal{E})$ and $H^3_*(\mathcal{E})$ that $\mathcal{E}$ can be expressed as the homology of a monad of the form

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{P} \oplus \mathcal{L} \xrightarrow{\beta} \mathcal{B} \to 0,$$

where $\mathcal{A}$ and $\mathcal{B}$ split and have ranks equal to the number of minimal generators of the module $H^1_*(\mathcal{E})$, and where $\mathcal{L}$ is also split. Is it true that $\mathcal{L}$ is always zero?

We shall show that this is true in some cases mentioned below.

Suppose $\mathcal{L} \neq 0$. Then $\alpha = [\alpha', l_1]^\vee$ and $\beta = [\beta', l_2]$ where l_1, l_2 are matrices of polynomials (with no non-zero scalar entries). Let $\mathcal{G}$ be the kernel of β. Then by the usual argument,

$$H^2_*(\wedge^2 \mathcal{G}) = H^2_*(\mathcal{A} \otimes \mathcal{G}) = A \otimes N,$$

where $A = H^0_*(\mathcal{A})$ and $N = H^2_*(\mathcal{E})$.

From the sequence

$$0 \to \wedge^2 \mathcal{G} \to \wedge^2(\mathcal{P} \oplus \mathcal{L}) \to (\mathcal{P} \oplus \mathcal{L}) \otimes \mathcal{B} \to S_2 \mathcal{B} \to 0,$$

we get the exact complex of maps on cohomology groups

$$H^2_*(\wedge^2 \mathcal{G}) \to H^2_*(\wedge^2 \mathcal{P}) \oplus H^2_*(\mathcal{P} \otimes \mathcal{L}) \to H^2_*(\mathcal{P} \otimes \mathcal{B}).$$

In other words, we get the exact sequence

$$A \otimes N \xrightarrow{[*, l_1 \otimes 1]^\vee} H^2_*(\wedge^2 \mathcal{P}) \oplus (L \otimes N) \xrightarrow{[*, l_2 \otimes 1]} B \otimes N.$$

Lemma 7. *If $H^2_*(\mathcal{E})$ is Buchsbaum or has a Buchsbaum summand, then $\mathcal{L}$ equals 0.*

Proof. Recall that if $N = H^2_*(\mathcal{E})$ is Buchsbaum, it means that N is annihilated by multiplication by any positive degree form. So the map given by l_2 from $L \otimes N$ to $B \otimes N$ is the zero map. By exactness, $L \otimes N$ is in the image of $A \otimes N \xrightarrow{l_1} L \otimes N$, which however, is also the zero map. A similar contradiction is obtained if N has a Buchsbaum summand, by restricting the exact sequence to the Buchsbaum summand. □

With the aid of this result, we get the following amusing consequence.

Proposition 8. *If $\mathcal{E}$ is a non-split rank two bundle on $\mathbf{P}^4$ and if $H^2_*(\mathcal{E})$ is Buchsbaum, then $\mathcal{E}$ is a Horrocks-Mumford bundle.*

Proof. Suppose $H^2_*(\mathcal{E}) = N$ is equal to $\oplus_{i=1}^t k(a_i)$, thus equal to t copies of the module k in different shifts. By the last corollary, a monad for $\mathcal{E}$ then has the form

$$0 \to \mathcal{A} \to \oplus_{i=1}^t \Omega^2(a_i) \to \mathcal{B} \to 0,$$

where $\mathcal{A}$ and $\mathcal{B}$ have ranks equal to $3t-1$. Hence (as above) we get the exact sequence

$$A \otimes (\oplus_{i=1}^t k(a_i)) \to H^2_*(\wedge^2(\oplus_{i=1}^t \Omega^2(a_i))) \to B \otimes (\oplus_{i=1}^t k(a_i)).$$

The sum of the dimensions of the vector spaces on the sides is equal to $2t(3t-1)$, hence we must have $2t(3t-1)$ is greater than or equal to the dimension of the vector space, V, in the middle. Included in the middle vector space, V, are $\binom{t}{2}$ terms $H^2_*(\Omega^2(a_i) \otimes \Omega^2(a_j))$. We can estimate $H^2_*(\Omega^2 \otimes \Omega^2)$ as follows: from the exact sequence

$$0 \to \Omega^2 \otimes \Omega^2 \to 10\Omega^2(-2) \to 5\Omega^2(-1) \to \Omega^2 \to 0,$$

we see that $H^2(\Omega^2 \otimes \Omega^2(2))$ is 10 dimensional, and by Serre duality, so is $H^2(\Omega^2 \otimes \Omega^2(3))$.

Hence the dimension of V is at least $\binom{t}{2}20 = 10t(t-1)$.

It follows from the inequalities that $t = 1$ or $t = 2$.

The case $t = 1$ is ruled out by Corollary 17 in [R1]. Hence we look at the case $t = 2$.

Suppose $N = k(a) \oplus k(b)$, with a monad for $\mathcal{E}$ given by

$$0 \to \mathcal{A} \to \Omega^2(a) \oplus \Omega^2(b) \to \mathcal{B} \to 0.$$

If $\mathcal{E}$ has first Chern class c_1, comparing the monad and its dual gives us that

1. $\mathcal{A}^\vee = \mathcal{B}(-c_1)$,
2. $\Omega^2(5-a) \oplus \Omega^2(5-b) = \Omega^2(a-c_1) \oplus \Omega^2(b-c_1)$ (since ${\Omega^2}^\vee = \Omega^2(5)$).

Hence if $a \leq b$ then $a - c_1 = 5 - b$.

In the case where $a = b$, without loss of generality, we assume they are both zero, and $c_1 = -5$. In the sequence

$$A \otimes (k \oplus k) \to H^2_*(\wedge^2(\Omega^2 \oplus \Omega^2)) \to B \otimes (k \oplus k),$$

we know that the vector space in the middle has dimension at least 10 in degrees 2 and 3. It follows that if $A = \oplus_{i=1}^5 S(e_i)$, $B = \oplus_{i=1}^5 S(-e_i-5)$, then the ten integers $e_i, -e_i-5$ are concentrated at the values $-2, -3$. In the monad for $\mathcal{E}$, since Ω^2 has no sections in degree 2, each $e_i \leq -3$. It follows that each $e_i = -3$, hence giving the monad of a Horrocks-Mumford bundle ([HM]).

In the case where $a < b$, without loss of generality, assume $a = 0$. Repeating the argument, we conclude that the twenty integers $e_i, e_i + b, -e_i + c_1, -e_i + c_1 + b$ are concentrated at the values $b-2, b-3$. This is quite impossible: for example e_i would have to be the smaller value $b-3$ since $b > 0$ and then $b = 1$. So $e_i = -2$. But it is impossible to have a monad of this form since only $\Omega^2(b)$ in $\Omega^2 \oplus \Omega^2(b)$ has sections in degree 2, negating the possibility of an inclusion of bundles on the left. Thus this case cannot occur. □

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