Chapter VI
Products and Quotients

1. Introduction

In Chapter III we defined the product $X \times Y$ of two topological spaces and considered some of the simple properties of products. (*See Examples III.5.10-5.12 and Exercise IIIE20.*) The properties we explored hold equally well for products of any finite number of spaces $X_1 \times ... \times X_n$. For example, the product of two compact spaces is compact, so a simple induction argument shows that the product of any finite number of compact spaces is compact. Now we turn our attention to infinite products which will lead us to some very nice theorems. For example, infinite products will eventually help us decide which topological spaces are metrizable.

2. Infinite Products and the Product Topology

The set $X \times Y$ was defined as $\{(x, y) : x \in X , y \in Y\}$. How can we define an "infinite product" set $X = \prod\{X_\alpha : \alpha \in A\}$? Informally, we want to say something like

$$X = \prod\{X_\alpha : \alpha \in A\} = \{(x_\alpha) : x_\alpha \in X_\alpha\}$$

so that a point x in the product consists of "coordinates" x_α chosen from the X_α's. But what exactly does a symbol like $x = (x_\alpha)$ mean if there are "uncountably many coordinates?"

We can get an idea by first thinking about a countable product. For sets $X_1, X_2, ..., X_n$,... we can informally define the product set as a certain set of sequences: $X = \prod_{n=1}^{\infty} X_n = \{(x_n) : x_n \in X_n\}$. But if we want to be careful about set theory, then a legal definition of X should have the form $X = \{(x_n) \in U : x_n \in X_n\}$. From what "pre-existing set" U will the sequences in X be chosen? The answer is easy: given the sets X_n, the ZF axioms guarantee that the set $(\bigcup_{n=1}^{\infty} X_n)^{\mathbb{N}}$ exists. Then

$$X = \prod_{n=1}^{\infty} X_n = \{x \in (\bigcup_{n=1}^{\infty} X_n)^{\mathbb{N}} : x(n) = x_n \in X_n\}.$$

Thus the elements of $\prod_{n=1}^{\infty} X_n$ are certain functions (sequences) defined on the index set $\mathbb{N}$. This idea generalizes naturally to any product.

Definition 2.1 Let $\{X_\alpha : \alpha \in A\}$ be a collection of sets. We define the product set $X = \prod\{X_\alpha : \alpha \in A\} = \{x \in (\bigcup X_\alpha)^A : x(\alpha) \in X_\alpha\}$. The X_α's are called the factors of X. For each α, the function $\pi_\alpha : \prod\{X_\alpha : \alpha \in A\} \to X_\alpha$ defined by $\pi_\alpha(x) = x_\alpha$ is called the α^{th}-projection map. For $x \in X$, we write more informally $x_\alpha = x(\alpha) =$ the α^{th}-coordinate of x and write $x = (x_\alpha)$.

Caution: the index set A might not be ordered. So even though we use the informal notation $x = (x_\alpha)$, such phrases as "the first coordinate of x," "the next coordinate in x after x_α," and "the coordinate in x preceding x_α" may not make sense. The notation (x_α) is handy but can lead you into errors if you're not careful.

By definition, a point x in $\prod\{X_\alpha : \alpha \in A\}$ is a function that "chooses" a coordinate x_α from each set in the collection $\{X_\alpha : \alpha \in A\}$. To say that such a "choice function" x must exist if all the X_α's are nonempty is precisely the Axiom of Choice. (*See the discussion following Theorem I.6.8.*)

Theorem 2.2 The Axiom of Choice (AC) is equivalent to the statement that every product of nonempty sets is nonempty.

Note: In ZF set theory, certain special products can be shown to be nonempty without using AC. For example, if $X_n = \mathbb{N}$, then $\prod_{n=1}^{\infty} X_n = \{x \in \mathbb{N}^{\mathbb{N}} : x(n) \in \mathbb{N}\} = \mathbb{N}^{\mathbb{N}}$. Without using AC, we can precisely describe a point (= function) in the product $-$ for example, the identity function $i = \{(m,n) \in \mathbb{N} \times \mathbb{N} : m = n\}$ $-$ so $\mathbb{N}^{\mathbb{N}} = \prod_{n=1}^{\infty} X_n \neq \emptyset$. Can you give other similar examples?

We will often write $\prod\{X_\alpha : \alpha \in A\}$ as $\prod_{\alpha \in A} X_\alpha$. If the indexing set A is clearly understood, we may simply write $\prod X_\alpha$.

Example 2.3

1) If $A = \emptyset$, then $\prod\{X_\alpha : \alpha \in A\} = \{(\bigcup_{\alpha \in A} X_\alpha)^A : x(\alpha) \in X_\alpha\} = \{\emptyset\}$.

2) Suppose $X_{\alpha_0} = \emptyset$ for some $\alpha_0 \in A$. Then $\prod_{\alpha \in A} X_\alpha = \{(\bigcup_{\alpha \in A} X_\alpha)^A : x(\alpha) \in X_\alpha\}$. But $x(\alpha_0) \in X_{\alpha_0}$ is impossible so $\prod_{\alpha\alpha \in A} X_\alpha = \emptyset$.

3) Strictly speaking, we now have two different definitions for a finite product $X_1 \times X_2$:

i) $X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1,\ x_2 \in X_2\}$ (a set of ordered pairs)

ii) $X_1 \times X_2 = \{x \in (X_1 \cup X_2)^{\{1,2\}} : x(i) \in X_i\}$ (a set of functions)

But there is an obvious way to regard these two sets as "the same": the ordered pair (x_1, x_2) corresponds to the function $x = \{(1, x_1), (2, x_2)\} \in (X_1 \cup X_2)^{\{1,2\}}$.

4) If $A = \mathbb{N}$, then $\prod\{X_n : n \in \mathbb{N}\} = \prod_{n=1}^{\infty} X_n = \{x \in (\bigcup_{n=1}^{\infty} X_n)^{\mathbb{N}} : x_n \in X_n\}$ $= \{(x_1, x_2, ..., x_n, ...,) : x_n \in X_n\} =$ the set of all sequences (x_n) where $x_n \in X_n$.

5) Suppose the X_α's are identical, say $X_\alpha = Y$ for all $\alpha \in A$. Then $\prod\{X_\alpha : \alpha \in A\}$ $= \{x \in (\bigcup_{\alpha \in A} X_\alpha)^A : x(\alpha) \in X_\alpha\} = \{x \in Y^A : x_\alpha \in Y\} = Y^A$. If $|A| = m$, we will sometimes write this product simply as $Y^m =$ "the product of m copies of Y" because the number of factors m is often more important than the specific index set A.

6) Discuss: is the equation $\prod_{i \in I} A_i \cap \prod_{j \in J} B_j = \prod_{(i,j) \in I \times J} (A_i \cap B_j)$ always true? sometimes true? never true?

Now that we have a definition of the set $\prod\{X_\alpha : \alpha \in A\}$, we can think about a product topology. We begin by recalling the definition and a few basic facts about the "weak topology." (*See Example III.8.6.*)

Definition 2.4 Let X be a set. For each $\alpha \in A$, suppose $(X_\alpha, \mathcal{T}_\alpha)$ is a topological space and that $f_\alpha : X \to X_\alpha$. The weak topology on X generated by the collection $\mathcal{F} = \{f_\alpha : \alpha \in A\}$ is the smallest topology on X that makes all the f_α's continuous.

Certainly, there is at least one topology on X that makes all the f_α's continuous: the discrete topology. Since the intersection of a collection of topologies on X is a topology (*why?*), the weak topology exists – we can describe it "from the top down" as $\bigcap\{\mathcal{T}$: $\mathcal{T}$ is a topology on X that makes all the f_α's continuous$\}$.

However, this efficient description of the weak topology doesn't give us a useful idea about what sets are open. Usually it is more useful to describe the weak topology on X "from the bottom up." To make all the f_α's continuous it is necessary and sufficient that

for each $\alpha \in A$ and for each open set $U_\alpha \subseteq X_\alpha$, the set $f_\alpha^{-1}[U_\alpha]$ must be open.

Therefore the weak topology $\mathcal{T}$ is the smallest topology that contains all such sets $f_\alpha^{-1}[U_\alpha]$, and that is the topology for which $\mathfrak{S} = \{f_\alpha^{-1}[U_\alpha] : \alpha \in A,\ U_\alpha$ open in $X_\alpha\}$ is a subbase. (*See Example III.8.6.*)

Therefore a base for the weak topology consists of all finite intersections of sets from $\mathfrak{S}$. A typical basic open set has form $f_{\alpha_1}^{-1}[U_{\alpha_1}] \cap f_{\alpha_2}^{-1}[U_{\alpha_2}] \cap ... \cap f_{\alpha_n}^{-1}[U_{\alpha_n}]$ where each $\alpha_i \in A$ and each U_{α_i} is open in X_{α_i}. To cut down on symbols, we will use a special notation for these subbasic and basic open sets: we will write

$< U_\alpha > \ = f_\alpha^{-1}[U_\alpha] \ =$ a typical subbasic open set, and then
$< U_{\alpha_1} > \ \cap \ < U_{\alpha_2} > \ \cap ... \cap \ < U_{\alpha_n} > \ = U$ for a typical basic open set.
We then abbreviate further as $U = \ < U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n} >$.
So $x \in U = \ < U_{\alpha_1}, U_{\alpha_2}, ...U_{\alpha_n} >$ iff $f_{\alpha_i}(x) \in U_{\alpha_i}$ for each $i = 1, ..., n$.

This notation is not standard—but it should be because it's very handy. You should verify that to get a base for the weak topology $\mathcal{T}$ on X, it is sufficient to use only the sets $< U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n} >$ where each U_{α_i} is a basic (or even subbasic) open set in X_{α_i}.

Example 2.5 Suppose $A \subseteq (X, \mathcal{T})$ and let $i : A \to X$ be the inclusion map $i(a) = a$. Then the subspace topology on A is the same as the weak topology generated by $\mathcal{F} = \{i\}$. To see this, just notice that a base for the weak topology is $\{i^{-1}[U] : U$ open in $X\}$, and the sets $i^{-1}[U] = U \cap A$ are exactly the open sets in the subspace topology.

The following theorem tells us that a map f into a space X with the weak topology is continuous iff each composition $f_\alpha \circ f$ is continuous.

Theorem 2.6 Suppose $f : Z \to X$, where Z is a topological space and X has the weak topology generated by maps $f_\alpha : X \to (X_\alpha, \mathcal{T}_\alpha)$ ($\alpha \in A$). Then f is continuous if and only if $f_\alpha \circ f : Z \to X_\alpha$ is continuous for every α.

Proof If f is continuous, then each composition $f_\alpha \circ f$ is continuous. Conversely, suppose each $f_\alpha \circ f$ is continuous. To show that f is continuous, it is sufficient to show that $f^{-1}[V]$ is open in Z whenever V is a subbasic open set in X (*why?*). So let $V = \ < U_\alpha >$ with U_α open in X_α. Then $f^{-1}[V] = f^{-1}[f_\alpha^{-1}[U_\alpha]] = (f_\alpha \circ f)^{-1}[U_\alpha]$, which is open because $f_\alpha \circ f$ is continuous. •

Definition 2.7 For each $\alpha \in A$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a topological space. The product topology $\mathcal{T}$ on the set $\prod X_\alpha$ is the weak topology generated by the collection of projection maps $\mathcal{F} = \{\pi_\alpha : \alpha \in A\}$.

The product topology is sometimes called the "Tychonoff topology." *We always assume that a product $\prod X_\alpha$ of topological spaces has the product topology unless some other topology is explicitly stated.*

Because the product topology is a weak topology, a subbase consists of all sets of form $< U_\alpha > = \pi_\alpha^{-1}[U_\alpha]$, where U_α is open in X_α. A base, then, consists of all possible finite intersections of these sets :

$$< U_{\alpha_1} > \cap ... \cap < U_{\alpha_n} > = \pi_{\alpha_1}^{-1}[U_{\alpha_1}] \cap ... \cap \pi_{\alpha_n}^{-1}[U_{\alpha_n}] \qquad (n \in \mathbb{N})$$

$$= \{x \in \prod X_\alpha : x_{\alpha_i} \in U_{\alpha_i} \text{ for each } i = 1, ..., n\}$$

$$= \prod_{\alpha \in A} U_\alpha \ \underline{\text{where}}\ U_\alpha = X_\alpha \text{ for } \alpha \neq \alpha_1, \alpha_2, ..., \alpha_n \quad (*)$$

It is sufficient to use only U_α's which are basic (or even subbasic) open sets in X_α. Why?

A basic open set in $\prod X_\alpha$ "depends on only finitely many coordinates" in the following sense:

> $x \in < U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n} >$ iff x satisfies the finitely many restrictions $x_{\alpha_i} \in U_{\alpha_i}$. The (basic) open sets containing x are what we use to describe "closeness" to x, so we can say informally that in the product topology "closeness depends on only finitely many coordinates."

If the index set A is finite, then the condition in (*) is satisfied automatically, and a base for the product topology is the set of all "open boxes" $\prod_{\alpha \in A} U_\alpha$:

$$\prod_{\alpha \in A} U_\alpha = \prod_{i=1}^n U_i = < U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n} > = U_1 \times U_2 \times ... \times U_n$$

Thus, when A is finite, Definition 2.7 agrees with the earlier definition for product topologies in Chapter III (Example 5.11).

You might not have expected Definition 2.7. A "first guess" to define a topology on products might have been to use all boxes $\prod_{\alpha \in A} U_\alpha$ (U_α open in X_α) rather than the more restricted collection in (*). As just noted, that would be equivalent to Definition 2.7 for finite products, but not for infinite products. One can define a topology on the set $\prod_{\alpha \in A} X_\alpha$ using all boxes of the form $\prod_{\alpha \in A} U_\alpha$ as a base – an alternate topology called the box topology that contains, in general, many more open sets than the product topology because of the omission of the restriction on the U_α's in (*). We will try to indicate, below, why our definition of the product topology is the "right" one to use.

Theorem 2.8 Each projection $\pi_\alpha : \prod X_\alpha \to X_\alpha$ is continuous and open, and π_α is onto if $\prod X_\alpha \neq \emptyset$. A function $f : Z \to \prod X_\alpha$ is continuous if and only if $\pi_\alpha \circ f : Z \to X_\alpha$ is continuous for every α.

Proof Each π_α is onto if the product is nonempty (*why?*). By definition, the product topology makes all the π_α's continuous. To show that π_α is open, it is sufficient to show that the image of a basic open set $U = < U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n} >$ is open.

This is clear if $U = \emptyset$, and

for $U \neq \emptyset$, we get $\pi_\alpha[U] = \pi_\alpha[\ < U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n} >\ ] = \begin{cases} U_{\alpha_i} & \text{if } \alpha = \alpha_i \\ X_\alpha & \text{if } \alpha \neq \alpha_1, ..., \alpha_n \end{cases}$..

Finally, f is continuous iff each composition $\pi_\alpha \circ f$ is continuous because the product has the weak topology generated by the projections (Theorem 2.6). •

Generally, projection maps are <u>*not*</u> *closed. For example, $F = \{(x, y) \in \mathbb{R}^2 : y = \frac{1}{x},\ x > 0\}$ is closed in $\mathbb{R} \times \mathbb{R}$, but its projection $\pi_1[F]$ is not closed in $\mathbb{R}$.)*

Example 2.9

1) A subbasic open set in $\mathbb{R} \times \mathbb{R}$ has form $\pi_1^{-1}[U] = U \times \mathbb{R}$ or $\pi_2^{-1}[V] = \mathbb{R} \times V$, where U and V are open in $\mathbb{R}$. (*We still get a subbase if we only use open* <u>*intervals*</u> *U, V in $\mathbb{R}$.*). Then basic open sets have form $(U \times \mathbb{R}) \cap (\mathbb{R} \times V) = U \times V$. Therefore the product topology on $\mathbb{R} \times \mathbb{R}$ is the usual topology on $\mathbb{R}^2$.

The function $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by $f(t) = (t^2, \sin t^2)$ is continuous because the compositions $(\pi_1 \circ f)(t) = t^2$ and $(\pi_2 \circ f)(t) = \sin t^2$ are both continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

2) Let $X = \mathbb{N}^{\aleph_0}$ = "the product of countably many copies of $\mathbb{N}$." The singleton sets $\{n\}$ are a basis for $\mathbb{N}$. A base for the product topology consists of all sets $U = \prod U_n$ where finitely many U_n's are singletons and all the others are equal to $\mathbb{N}$. Each basic open set U is infinite (*in fact $|U| = c$ – why?*), so X has no isolated points and therefore X is not discrete. (*In fact, $X \simeq \mathbb{P}$*). By similar reasoning, an infinite product of discrete spaces, each with more than 1 point, is <u>not discrete</u>.

For each $x = (k_1, k_2, ..., k_n, ...) \in X$, the set $\{x\} = \prod_{n=1}^{\infty} \{k_n\}$ is open <u>in the box topology</u> so, in contrast, the box topology on X <u>is</u> the discrete topology. (*For a finite product, the box and product topologies care the same: a* <u>*finite*</u> *product of discrete spaces* <u>*is*</u> *discrete.*)

3) Consider $X = \prod\{X_r : r \in \mathbb{R}\}$ where each $X_r = \mathbb{R}$. By definition of product, each point in X is a function $f : \mathbb{R} \to \mathbb{R}$. In other words, $X = \mathbb{R}^{\mathbb{R}}$ and for each r, $\pi_r(f) = f(r)$. As basic open sets in X, we can use sets $U = \ < U_{r_1}, U_{r_2}, ..., U_{r_n} >$, where the U_{r_i}'s are open intervals in $\mathbb{R}$. Then $f \in U$ if and only if $f(r_i) \in U_{r_i}$ for each $i = 1, ..., n$. If g is <u>also</u> in U, then f and g are "close" at the finitely many points r_i – in the sense that $f(r_i)$ and $g(r_i)$ are both in the interval U_{r_i}. If, for example, the U_{r_i}'s each have diameter less than ϵ, then $|f(r_i) - g(r_i)| < \epsilon$ for each $i = 1, ..., n$. *Of course, this is much weaker than saying f and g are "uniformly" close.*

Why is the product topology the "correct" topology for set $\prod X_\alpha$? Of course there is no "right" answer, but a few observations should make it seem a good choice.

Example 2.10

1) For finite products, the box and product topologies are exactly the same. When it comes to <u>infinite</u> products, there's no obvious reason to favor the box topology or the product topology. Moreover, if one of them seems more natural, then at least we should be cautious: our intuition, after all, is only comfortable with finite sets, and we always run risks when we apply naive intuition to infinite collections.

2) Consider $(0,1)$: for two points $x = 0.x_1 \dots x_n \dots$ and $a = 0.a_1 \dots a_n \dots$, it will be true that "$x$ is close to a" when x and a agree in the first n decimal places (for a sufficiently large n). Roughly speaking, "closeness" depends on only finitely many decimal places ("coordinates").

Now consider the Hilbert cube $H = \prod_{n=1}^{\infty}[0, \frac{1}{n}] = [0,1] \times [0, \frac{1}{2}] \times \dots \times [0, \frac{1}{n}] \times \dots \subseteq \ell_2$, where ℓ_2 has its usual metric, d (*see Example II.2.6.6 and Exercise II.E10*). Suppose $x, a \in H$ and let $\epsilon > 0$. What condition on x will guarantee that $d(x,a) < \epsilon$?

Pick N so that $\sum_{i=N+1}^{\infty} \frac{1}{i^2} < \frac{\epsilon^2}{2}$. If $(x_i - a_i)^2 < \frac{\epsilon^2}{2N}$ for each $i = 1, \dots, N$, then we have $d(x,a) = (\sum_{i=1}^{N}(x_i - a_i)^2 + \sum_{i=N+1}^{\infty}(x_i - a_i)^2)^{1/2} < (N \cdot \frac{\epsilon^2}{2N} + \frac{\epsilon^2}{2})^{1/2} = \epsilon$. Here, in the natural metric topology on the product H, we see that we can achieve "x close to a" by requiring "closeness" in just finitely many coordinates $1, \dots, N$. This is just what the product topology does. In fact, the product topology on H turns out to be the topology $\mathcal{T}_d$.

A handy "rule of thumb" that has proved true every time I've used it is that if a topology on a product set is such that "closeness depends on only finitely many coordinates," then that topology is the product topology.

3) From a very pragmatic point of view, the product topology appears much more manageable. To "get your mind around" a basic open set $U = < U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n} >$ in the product topology, you only need to think about finitely many sets $U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$; but in the box topology, thinking about U requires taking into account all the U_α's in $U = \prod_{\alpha \in \in} U_\alpha$ – and there may be uncountably many different U_α's.

4) The bottom line, however, is this: a mathematical definition justifies itself by the fruit it bears. The definition of the product topology will lead to some beautiful theorems. Using the product topology, for example, we will see that compact Hausdorff spaces are topologically nothing other than the closed subspaces of cubes $[0,1]^m$ (where m might be infinite). For the time being, you will need to accept that things work out nicely down the road, and that by contrast, the box topology turns out to be rather ill-behaved. (*See Exercise E11.*)

As a simple example of such nice behavior, the following theorem is exactly what one would hope for – and the proof depends on having the "correct" topology on the product. The theorem says that convergence of sequences in a product is "coordinatewise convergence" : that is, in a product, $(x_n) \to x$ iff for all α , the α^{th} coordinate of x_n converges (in X_α) to the α^{th} coordinate of x. For that reason, the product topology is sometimes called the "topology of coordinatewise convergence."

Theorem 2.11 Suppose (x_n) is a sequence in $X = \prod\{X_\alpha : \alpha \in A\}$. Then $(x_n) \to x \in X$ iff $(\pi_\alpha(x_n)) \to \pi_\alpha(x)$ in X_α for all $\alpha \in A$.

Proof If $(x_n) \to x$, then $(\pi_\alpha(x_n)) \to \pi_\alpha(x)$ because each π_α is continuous.

Conversely, suppose $(\pi_\alpha(x_n)) \to \pi_\alpha(x)$ in X_α for each α and consider any basic open set $U = < U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_k} >$ that contains $x = (x_\alpha)$. For each $i = 1, \dots, k$, we have $x_{\alpha_i} \in U_{\alpha_i}$. Since $(\pi_{\alpha_i}(x_n)) \to \pi_{\alpha i}(x)$, we have $\pi_{\alpha_i}(x_n) \in U_{\alpha_i}$ for $n \geq$ some N_i. Let $N = \max\{N_1, \dots, N_k\}$. Then for $n \geq N$ we have $\pi_{\alpha_i}(x_n) \in U_{\alpha_i}$ for every $i = 1, \dots, k$. This means that $x_n \in U$ for $n \geq N$, so $(x_n) \to x$. •

In the proof, N is the max of a finite set. If X has the box topology, the basic open set $U = \prod U_\alpha$ might involve infinitely many open sets $U_\alpha \neq X_\alpha$. For each such α, we could pick an $N_\alpha \in \mathbb{N}$, just as in the proof. But the set of all N_α's might not have a max, N, and the proof would collapse. Can you create a specific example with the box topology where this happens?

Example 2.12 Consider $\mathbb{R}^\mathbb{R} = \prod\{X_r : r \in \mathbb{R}\}$, where $X_r = \mathbb{R}$. Each point f in the product is a function $f : \mathbb{R} \to \mathbb{R}$. Suppose that (f_n) is a sequence of points in $\mathbb{R}^\mathbb{R}$. By Theorem 2.11, $(f_n) \to f$ iff $(f_n(r)) \to f(r)$ for each $r \in \mathbb{R}$. With the product topology, convergence of a sequence of functions in $\mathbb{R}^\mathbb{R}$ is called (in analysis) pointwise convergence. *Question: if $\mathbb{R}^\mathbb{R}$ is given the box topology, is convergence of a sequence (f_n) simply uniform convergence (as defined in analysis)?*

The following "theorem" is stated loosely. You can easily create variations. Any reasonable version of the statement is probably true.

Theorem 2.13 Topological products are associative in any "reasonable" sense: for example, if the index set A is written as $A = B \cup C$ where B and C are disjoint, then

$$\begin{array}{ccccc} \prod\{X_\alpha : \alpha \in A\} & \simeq & \prod\{X_\beta : \beta \in B\} & \times & \prod\{X_\gamma : \gamma \in C\} \\ || & & || & & || \\ X & & Y & \times & Z \end{array}$$

Proof A point $x \in \prod\{X_\alpha : \alpha \in A\}$ is a function $x : A \to \bigcup_{\alpha \in A} X_\alpha$. Define $f : X \to Y \times Z$ by $f(x) = (x|B, x|C)$. Clearly f is one-to-one and onto.

f is a mapping of X into a product $Y \times Z$, so f is continuous iff $\pi_1 \circ f$ and $\pi_2 \circ f$ are both continuous. But $\pi_1 \circ f$ is also a map into a product: $\pi_1 \circ f : X \to Y = \prod\{X_\beta : \in B\}$. So $\pi_1 \circ f$ is continuous if and only if $\pi_\beta \circ \pi_1 \circ f : \prod\{X_\alpha : \alpha \in A\} \to X_\beta$ is continuous for all $\beta \in B$. This is true because $\pi_\beta \circ \pi_1 \circ f = \pi_\beta : \prod\{X_\alpha : \alpha \in A\} \to X_\beta$. The proof that $\pi_2 \circ f$ is continuous is completely similar.

$f^{-1} : Y \times Z \to X = \prod\{X_\alpha : \alpha \in A\}$ is given by $x = f^{-1}(y, z) = y \cup z$ ($=$ *the union of two functions*), and f^{-1} is continuous iff $\pi_\alpha \circ f^{-1} : Y \times Z \to X_\alpha$ is continuous for each $\alpha \in A = B \cup C$. To check this, first suppose $\alpha \in B$: then $\pi_\alpha \circ f^{-1}(y, z) = \pi_\alpha(x)$, where $x = y \cup z$. Since $\alpha \in B$, $x_\alpha = (\pi_\alpha \circ \pi_1)(y, z)$, so $\pi_\alpha \circ f^{-1} = \pi_\alpha \circ \pi_1$, which is continuous. The case where $\alpha \in C$ is completely similar.

Therefore f is a homeomorphism. •

The question of topological commutativity for products only makes sense when the index set A is ordered in some way. But even then: if we view a product as a collection of functions, the question of commutativity is trivial – the question reduces to the fact that set theoretic unions are commutative. For example,

$$\begin{aligned} X_1 \times X_2 &= \{x \in (X_1 \cup X_2)^{\{1,2\}} : x(i) \in X_i \text{ for } i = 1, 2\} \\ &= \{x \in (X_2 \cup X_1)^{\{1,2\}} : x(i) \in X_i \text{ for } i = 1, 2\} = X_2 \times X_1. \end{aligned}$$

So viewed as sets of functions, $X_1 \times X_2$ and $X_2 \times X_1$ are exactly the same set! The same observation applies to any product viewed as a collection of functions.

But we might look at an ordered product in another way: for example, thinking of $X_1 \times X_2$ and $X_2 \times X_1$ as sets of ordered pairs. Then generally $X_1 \times X_2 \neq X_2 \times X_1$. From that point of view, the topological spaces $X_1 \times X_2$ and $X_2 \times X_1$ are not literally identical, but there is a homeomorphism between them: $f(x_1, x_2) = (x_2, x_1)$. So the products are topologically identical. We can make a similar argument whenever the order of factors is "commuted" by permuting the index set.

The general rule of thumb is that "whenever it makes sense, topological products are commutative."

Exercise 2.14 X^m denotes the product of m copies of the space X. Prove that $(X^m)^n$ is homeomorphic to X^{mn}. (*Hint: The bijection ϕ in the proof of Theorem I.14.7 is a homeomorphism.*)

Notice that "cancellation properties" may not be true. For example, $\mathbb{N} \times \{0\}$ and $\mathbb{N} \times \{0,1\}$ are homeomorphic (both are countable discrete spaces) but topologically you can't "cancel the $\mathbb{N}$" : $\{0\}$ is not homeomorphic to $\{0,1\}$!

Here are a few results which are quite simple but very handy to remember**.** The first states that singleton factors are topologically irrelevant in a product.

Lemma 2.15 $\prod_{\alpha\in A} X_\alpha \times \prod_{\beta\in B}\{p_\beta\} \simeq \prod_{\alpha\in A} X_\alpha$

Proof $\prod_{\beta\in B}\{p_\beta\}$ is itself a one-point space $\{p\}$, so we only need to prove that $\prod_{\alpha\in A} X_\alpha \times \{p\} \simeq \prod_{\alpha\in A} X_\alpha$. The map $f(x, p) = x$ is clearly a homeomorphism. •

Lemma 2.16 Suppose $\prod_{\alpha\in A} X_\alpha \neq \emptyset$. For any $B \subseteq A$, $\prod_{\alpha\in B} X_\alpha$ is homeomorphic to a subspace Z of $\prod_{\alpha\in A} X_\alpha$ – that is, $\prod_{\alpha\in B} X_\alpha$ can be embedded in $\prod_{\alpha\in A} X_\alpha$. In fact, if all the X_α's are T_1-spaces, then $\prod_{\alpha\in B} X_\alpha$ is homeomorphic to a <u>closed</u> subspace Z of $\prod_{\alpha\in A} X_\alpha$.

Proof Pick a point $p = (p_\alpha) \in \prod_{\alpha\in A} X_\alpha$. Then by Lemma 2.15,

$$\prod_{\alpha\in B} X_\alpha \simeq \prod_{\alpha\in B} X_\alpha \times \prod_{\alpha\in A-B}\{p_\alpha\} = Z \subseteq \prod_{\alpha\in A} X_\alpha$$

Now suppose all the X_α's are T_1. If $y \in (\prod_{\alpha\in A} X_\alpha) - Z$, then for some $\gamma \in A - B$, $y_\gamma \neq p_\gamma$. Since X_γ is a T_1 space, there is an open set U_γ in X_γ that contains y_γ but not p_γ. Then $y \in\ < U_\gamma >$ and

$$< U_\gamma > \cap (\prod_{\alpha\in B} X_\alpha \times \prod_{\alpha\in A-B}\{p_\alpha\}) = \emptyset.$$

Therefore Z is closed in $\prod_{\alpha\in A} X_\alpha$. •

Note: 1) Assume each $X_\alpha = \mathbb{R}$. Go through the preceding proof step-by-step when $A = \{1, ..., k\}$ and when $A = \mathbb{N}$ and

2) In the case $B = \{\alpha_0\}$, Lemma 2.16 says that each factor X_{α_0} is homeomorphic to subspace of $\prod_{\alpha\in A} X_\alpha$ (a closed subspace if all the X_α's are T_1).

3) Caution: Lemma 2.16 does <u>not</u> say that if all the X_α's are T_1, then <u>every</u> copy of X_{α_0} embedded in $\prod_{\alpha\in A} X_\alpha$ is closed: only that there <u>exists</u> a closed homeomorphic copy. (It is very easy to show a copy of $\mathbb{R}$ embedded in $\mathbb{R}^2$ that is not closed in $\mathbb{R}^2$, for example ... ?)

Lemma 2.17 Suppose $X = \prod\{X_\alpha : \alpha \in A\} \neq \emptyset$. Then X is a Hausdorff space (or, T_1-space) if and only if every factor X_α is a Hausdorff space (or, T_1-space).

Proof, for T_2 Suppose all the X_α's are Hausdorff. If $x \neq y \in X$, then $x_{\alpha_0} \neq y_{\alpha_0}$ for some α_0. Pick disjoint open sets U_{α_0} and V_{α_0} in X_{α_0} containing x_{α_0} and y_{α_0}. Then $< U_{\alpha_0} >$ and $< V_{\alpha_0} >$ are disjoint (basic) open sets in $\prod X_\alpha$ that contain x and y, so X is Hausdorff.

Conversely, suppose $X \neq \emptyset$. By Lemma 2.16, each factor X_α is homeomorphic to a subspace of X. Since a subspace of a Hausdorff space is Hausdorff (*why?*), each X_α is a Hausdorff space. •

Exercise 2.18 Prove Lemma 2.17 if "Hausdorff" is replaced by "T_1." (*The proof is similar but easier.*)

Theorem 2.19 The product of countably many two-point discrete spaces is homeomorphic to the Cantor set C.

Proof We will show that $\prod_{n=1}^\infty X_n \simeq C$, where each $X_n = \{0, 2\}$.

To construct C we defined, for each sequence $x = (x_1, x_2, ..., x_n, ...) \in \{0, 2\}^{\mathbb{N}}$, a descending sequence of closed sets $F_{x_1} \supseteq F_{x_1 x_2} \supseteq ... \supseteq F_{x_1 x_2 ... x_n} \supseteq ...$ in $[0, 1]$ whose intersection gave a unique point $p \in C$: $\{p\} = \bigcap_{n=1}^\infty F_{x_1 x_2 ... x_n}$ (*see Section IV.10*). For each n, we can write C as a union of 2^n disjoint clopen sets: $C = \bigcup_{(x_1, ..., x_n) \in \{0,2\}^n} (C \cap F_{x_1 x_2 ... x_n})$.

Define $f : C \to \prod_{n=1}^\infty X_n$ by $f(p) = x = (x_1, x_2, ..., x_n, ...)$. Clearly f is one-to-one and onto. To show that f is continuous at $p \in C$, it is sufficient to show that for each n, the function $\pi_n \circ f : C \to \{0, 2\}$ is continuous at p. Pick any $n \in \mathbb{N}$. For this n, there is a clopen set $U = C \cap F_{x_1 x_2 ... x_n}$ that contains p, and $(\pi_n \circ f)|U = x_n$. Thus, $\pi_n \circ f$ is constant on a neighborhood of p in C, so $\pi_n \circ f$ is continuous at p.

By Lemma 2.17, $\prod_{n=1}^\infty X_n$ is Hausdorff. Since f is a continuous bijection of a compact space onto a Hausdorff space, f is a homeomorphism (*why?*). •

Corollary 2.20 $\{0, 2\}^{\aleph_0}$ is compact.

This corollary is a very special case of the Tychonoff Product Theorem which states that any product of compact spaces is compact. The Tychonoff Product Theorem is much harder and will be proved in Chapter IX.

Corollary 2.21 The Cantor set is homeomorphic to a product of countably many copies of itself.

Proof By Exercise 2.14 above, $C^{\aleph_0} \simeq (\{0, 2\}^{\aleph_0})^{\aleph_0} \simeq \{0, 2\}^{\aleph_0 \cdot \aleph_0} \simeq \{0, 2\}^{\aleph_0} \simeq C$. The case of $C^n \simeq C$ for $n \in \mathbb{N}$ is similar. •

Example 2.22 Convince yourself that each assertion is true:

1) If X is the Sorgenfrey line, then $X \times X$ is the Sorgenfrey plane (*see Examples III.5.3 and III.5.4*).

2) Let S^1 be the unit circle in $\mathbb{R}^2$. Then $S^1 \times [0, 1]$ is homeomorphic to the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z \in [0, 1]\}$.

3) $S^1 \times S^1$ is homeomorphic to a torus (= "the surface of a doughnut").

Exercises

E1. Does it ever happen that $X \times \{0\}$ open in $X \times \mathbb{R}$? If so, what is a necessary and sufficient condition on X for this to happen?

E2. a) Suppose X and Y are topological spaces and that $A \subseteq X$, $B \subseteq Y$. Prove that $\text{int}_{X \times Y}(A \times B) = \text{int}_X A \times \text{int}_Y B$: that is, "the interior of the product is the product of the interiors." (*By induction, the same result holds for any* finite *product*.) Give an example to show that the statement may be false for infinite products.

b) Suppose $A_\alpha \subseteq X_\alpha$ for all $\alpha \in A$. Prove that in the product $X = \prod X_\alpha$,

$$\text{cl}\,(\prod A_\alpha) = \prod \text{cl}\, A_\alpha.$$

Note: When the A_α's are closed, this shows that $\prod A_\alpha$ is closed: so "any product of closed sets is closed." Can you see any plausible reason why products of closures are better behaved than products of interiors?

c) Suppose $X = \prod X_\alpha \neq \emptyset$ and that $A_\alpha \subseteq X_\alpha$. Prove that $\prod A_\alpha$ is dense in X iff A_α is dense in X_α for each α. *Note: Part c) implies that a finite product of separable spaces is separable, but it doesn't tell us whether or not an infinite product of separable spaces is separable: why not?*

d) For each α, let $q_\alpha \in X_\alpha$. Prove that $B = \{x \in \prod X_\alpha : x_\alpha = q_\alpha$ for all but at most finitely many $\alpha\}$ is dense in $\prod X_\alpha$. *Note: Suppose $X = \mathbb{R}^{\mathbb{N}} = \prod_{n \in \mathbb{N}} X_n$ where each $X_n = \mathbb{R}$. Suppose each q_n is chosen to be a rational – say $q_n = 0$. Then what does d) imply about $\mathbb{R}^{\mathbb{N}}$?*

e) Let $\alpha_0 \in A$. Prove that $Y_{\alpha_0} = \{x \in \prod X_\alpha : x_\alpha = p_\alpha$ for all $\alpha \neq \alpha_0\}$ is homeomorphic to X_{α_0}. *Note: So any factor of a product has a "copy" of itself inside the product in a "natural" way. For example, in $\mathbb{R}^n$, the set of points where all coordinates except the first are 0 is homeomorphic to the first factor, $\mathbb{R}$.*.

f) Give an example of infinite spaces X, Y, Z such that $X \times Y$ is homeomorphic to $X \times Z$ but Y is not homeomorphic to Z.

E3. Let X be a topological space and consider the "diagonal" Δ of $X \times X$:

$$\Delta = \{(x, x) : x \in X\} \subseteq X \times X.$$

a) Prove that Δ is closed in $X \times X$ if and only if X is Hausdorff.
b) Prove that Δ is open in $X \times X$ if and only if X is discrete.

E4. Suppose X is a Hausdorff space and that $X_\alpha \subseteq X$ for each $\alpha \in A$. Show that $Y = \bigcap\{X_\alpha : \alpha \in A\}$ is homeomorphic to a closed subspace of the product $\prod\{X_\alpha : \alpha \in A\}$.

E5. For each $n \in \mathbb{N}$, suppose D_n is a countable dense set in $(X_n, \mathcal{T}_n)$.

a) Prove that $D = \prod D_n$ is dense in $\prod X_n$.

b) Prove that $\prod X_n$ is separable. *Hint: Note that D might not be countable! But closeness in a product depends on only finitely many coordinates.*

E6. a) Suppose $a_{i,j} \in \{0, 2\}$ for all $i, j \in \mathbb{N}$. Prove that there exists a sequence (j_k) in $\mathbb{N}$ such that, for each i, $\lim_{k \to \infty} a_{i,j_k}$ exists. *Hint: "picture" the $a_{i,j}$ in an infinite matrix. For each fixed j, the "j-th column" of the matrix is a point in the Cantor set $C = \{0, 2\}^{\aleph_0}$.*

b) In $\mathbb{R}$, $\sum_{n=1}^{\infty} a_n'$ is called a <u>subseries</u> of $\sum_{n=1}^{\infty} a_n$ if for every n $\begin{cases} a_n' = a_n & \text{or} \\ a_n' = 0 \end{cases}$

Prove that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $S = \{s \in \mathbb{R} : s \text{ is the sum of a subseries of } \sum_{n=1}^{\infty} a_n\}$ is closed in $\mathbb{R}$. *Hint: "Absolute convergence" guarantees that every subseries converges. Each subseries $\sum_{n=1}^{\infty} a_n'$ can be associated in a natural way with a point $x \in \{0, 1\}^{\aleph_0}$. Consider the mapping $f : \{0, 1\}^{\aleph_0} \to \mathbb{R}$ given by $f(x) = \sum_{n=1}^{\infty} a_n' \in \mathbb{R}$. Must f be a homeomorphism?*

c) Suppose G is an open dense subset of the Cantor set C. Must $\operatorname{Fr} G$ be countable? *Hint: Consider $\{(a, b) \in C \times C : b \neq 0\}$.*

E7. Let $\mathbb{N}$ have the cofinite topology.

a) Does the product $\mathbb{N}^m$ have the cofinite topology? Does the answer depend on m?

b) Prove $\mathbb{N}^m$ is separable *Hint: When m is infinite, consider the simplest possible points in the product. Note: part b) implies that an arbitrarily large product of T_1 spaces with more than one point can be separable. However, that is false for Hausdorff spaces – see Theorem 3.8 later in this chapter.*

E8. We can define a topology on any set X by choosing a nonempty family of subsets $\mathcal{F}$ and defining closed sets to be all sets which can be written as an intersection of finite unions of sets from $\mathcal{F}$. $\mathcal{F}$ is called a subbase for the closed sets of X. (*This construction is "complementary" to generating a topology on X by using a collection of sets as a subbase for the open sets.*)

a) Verify that this procedure does define a topology on X.

b) Suppose X_α is a topological space. Give $\prod X_\alpha$ the topology for which collection of "closed boxes" $\mathcal{F} = \{\prod F_\alpha : F_\alpha \text{ closed in } X_\alpha\}$ is a subbase for the closed sets. Is this topology the product topology?

E9. Prove or disprove:

There exists a bijection $f : X = \{0,1\}^{\aleph_0} \to \mathbb{N}^{\aleph_0} = Y$ such that for all $x \in X$ and for all n, the first n coordinates of $y = f(x)$ are determined by the first m coordinates of x.

Here, m depends on n and x. More formally, we are asking whether there exists a bijection f such that:

$\forall x \in X \ \forall n \in \mathbb{N} \ \exists m \in \mathbb{N}$ *such that changing x_j for any $j > m$ does not change y_k for $k \leq n$*

Hint: Think about continuity and the definition of the product topology.

3. Productive Properties

We want to consider how some familiar topological properties behave with respect to products.

Definition 3.1 Suppose that, for each $\alpha \in A$, the space X_α has a certain property P. We say that

the property P is $\begin{cases} \underline{\text{productive}} & \text{if } \prod X_\alpha \text{ must have property } P \\ \underline{\text{countably productive}} & \text{if } \prod X_\alpha \text{ must have property } P \text{ when } A \text{ is countable} \\ \underline{\text{finitely productive}} & \text{if } \prod X_\alpha \text{ must have property } P \text{ when } A \text{ is finite} \end{cases}$

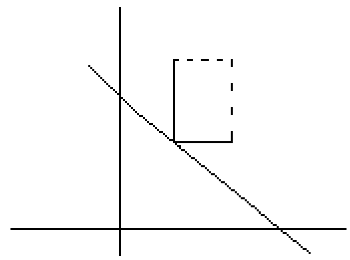
For example, Lemma 2.17 shows that the T_1 and T_2 properties are productive.

Some topological properties behave very badly with respect to products. For example, the Lindelöf property is a "countability property" of spaces, and we might expect the Lindelöf property to be countably productive. Unfortunately, this is not the case.

Example 3.2 The Lindelöf property is not finitely productive; in fact if X is a Lindelöf space, then $X \times X$ may not be Lindelöf. Let X be the set of real numbers with the topology for which a neighborhood base at a is $\mathcal{B}_a = \{[a, b) : a, b \in S,\ b > a\}$. (*Recall that X is called the Sorgenfrey Line: see Example III.5.3.*) We begin by showing that $\mathcal{S}$ is Lindelöf.

> It is sufficient to show that a collection $\mathcal{U}$ of <u>basic</u> open sets covering X has a countable subcover. Given such a cover $\mathcal{U}$, Let $\mathcal{V} = \{(a, b) :\ [a, b) \in \mathcal{U}\}$ and define $A = \bigcup \mathcal{V}$. For a moment, think of A as a subspace of $\mathbb{R}$ with its <u>usual</u> topology. Then A is Lindelöf (*why?*), and $\mathcal{V}$ is a covering of A by usual open sets, so there is a countable subfamily $\mathcal{V}'$ with $\bigcup \mathcal{V}'$ $= A$.
>
> Now replace the left endpoints of the intervals in $\mathcal{V}'$ to get $\mathcal{U}' = \{[a, b) : (a, b) \in \mathcal{V}'\} \subseteq \mathcal{U}$. If $\mathcal{U}'$ covers X we are done, so suppose $X - \bigcup \mathcal{U}' \neq \emptyset$. For each $x \in X - \bigcup \mathcal{U}'$, pick a set $[a, b)$ in $\mathcal{U}$ that contains x. In fact, x <u>must</u> be the left endpoint of $[a, b)$ – because if $x \in [a, b) \in \mathcal{U}$ and $x \neq a$, then $x \in \bigcup \mathcal{V} = \bigcup \mathcal{V}' \subseteq \bigcup \mathcal{U}'$. So, for each $x \notin \bigcup \mathcal{U}'$ we can pick a set $[x, b_x) \in \mathcal{U}$.
>
> If x and y are distinct points not in $\bigcup \mathcal{U}'$, then $[x, b_x)$ and $[y, b_y)$ must be disjoint (*why?*) and there can be at most countably many disjoint intervals $[x, b_x)$. So $\mathcal{U}' \cup \{[x, b_x) : x \notin \bigcup \mathcal{U}'\}$ is a countable subcollection of $\mathcal{U}$ that covers S.

However the Sorgenfrey plane $S \times S$ is <u>not</u> Lindelöf. If it were, then its closed subspace $D = \{(x, y) : x + y = 1\}$ would also be Lindelöf (Theorem III.7.10). But that is impossible since D is uncountable and discrete in the subspace topology. (*See the figure on the following page.*)



Fortunately, many other topological properties do play more nicely with products. Here are several topological properties P to which the "same" theorem applies. We combine these into one large theorem for efficiency.

Theorem 3.3 Suppose $X = \prod\{X_\alpha : \alpha \in A\} \neq \emptyset$. Let P be one of the properties "first countable," "second countable," "metrizable," or "completely metrizable." Then X has property P iff

1) all the X_α's have property P, and
2) at most countably many X_α's are <u>non</u>trivial (i.e., do not have the trivial topology)

For all practical purposes, this theorem is a statement about countable products because:

1) The nontrivial X_α's are the "interesting" factors, and 2) says there are only countably many of them. In practice, one hardly ever works with trivial spaces, and if we totally exclude trivial spaces from the discussion, then the theorem just states that X has property P iff X is a countable product of spaces with property P.

2) A nonempty T_1-space X_α has the trivial topology iff $|X_\alpha| = 1$. So, if we are concerned only with T_1-spaces (as is most often the case) the theorem says that X has property P iff all the X_α's have property P and all but countably many of the X_α's are "topologically irrelevant" singletons. Of course, in the cases that involve metrizability, the T_1 condition is automatically satisfied.

Proof <u>Throughout the proof</u>, let $B =$ the set of "interesting indices" $= \{\alpha \in A : X_\alpha$ is a nontrivial space$\}$. We begin with the case $P =$ "first countable."

<u>Suppose 1) and 2) hold</u> and $x \in X$. We need to produce a countable neighborhood base at x. For each $\alpha \in B$, let $\{U_\alpha^n : n \in \mathbb{N}\}$ be a countable open neighborhood base at $x_\alpha \in X_\alpha$. Let

$$\mathcal{B}_x = \{U : U \text{ is a finite intersection of sets of the form } < U_\alpha^n > ,\ \alpha \in B,\ n \in \mathbb{N}\}$$

Since B is countable, $\mathcal{B}_x$ is a countable collection of open sets containing x and we claim that $\mathcal{B}_x$ is a neighborhood base at $x \in X$. To see this, suppose $V = < V_{\alpha_1}, ..., V_{\alpha_k} >$ is a basic open set containing x. (*We may assume that all α_i's are in B : why?*) For each $i = 1, ..., k$, pick $U_{\alpha_i}^{n_i}$ so that $x_{\alpha_i} \in U_{\alpha_i}^{n_i} \subseteq V_{\alpha_i}$. Then $U = < U_{\alpha_1}^{n_1}, U_{\alpha_2}^{n_2}, ..., U_{\alpha_k}^{n_k} > \in \mathcal{B}_x$ and $x \in U \subseteq V$.

Conversely, suppose X is first countable. We need to prove that 1) and 2) hold.

$X \neq \emptyset$, so by Lemma 2.16 each X_α is homeomorphic to a subspace of X. Therefore each X_α is first countable, so 1) holds.

For 2) we prove the contrapositive: assuming B is uncountable, we find a point $x \in X$ at which there cannot be a countable neighborhood base. Pick any point $p = (p_\alpha) \in X$

> For each $\beta \in B$, pick an open set $O_\beta \subseteq X_\beta$ for which $\emptyset \neq O_\beta \neq X_\beta$. Choose $x_\beta \in O_\beta$ and $y_\beta \notin O_\beta$. Define $x = (x_\alpha) \in X$ using the coordinates
>
> $$x_\alpha = \begin{cases} x_\beta & \text{if } \alpha = \beta \in B \\ p_\alpha \in X_\alpha & \text{if } \alpha \notin B \end{cases}$$

Suppose $\mathcal{B}_x$ is a countable collection of neighborhoods of x and for each $N \in \mathcal{B}_x$ pick a basic open set U with $x \in U = \, < U_{\alpha_1}, ..., U_{\alpha_k} > \, \subseteq N$. There are only finitely many α_i's involved in the expression for each chosen U, and there are only countably many N's in $\mathcal{B}_x$. So, since B is uncountable, we can pick a $\beta \in B$ that is not one of the α_i's involved in the expression for any of the sets U that were picked. Then

> i) $< O_\beta >$ is an open set that contains x because $x_\beta \in O_\beta$
>
> ii) for all $N \in \mathcal{B}_x$, $x \in N \nsubseteq \, < O_\beta >$, so $\mathcal{B}_x$ cannot be a neighborhood base at x. To see this, define a point $w = (w_\alpha)$ by
>
> $$w_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \neq \beta \\ y_\beta & \text{if } \alpha = \beta \end{cases}$$
>
> Thus, w and x have the same coordinates except the β coordinate. For each $N \in B_x$, we picked $U = \, < U_{\alpha_1}, ..., U_{\alpha_k} > \, \subseteq N$. Since $x \in U$ and w has the same α_1,..., α_n coordinates as x, we also have $w \in U$. But $w \notin \, < O_\beta >$ because $w_\beta = y_\beta \notin O_\beta$. Therefore $U \nsubseteq \, < O_\beta >$, so $N \nsubseteq \, < O_\beta >$.

It's now easy to see that if the product $X = \prod X_\alpha$ has any of the other properties P, then conditions 1) and 2) must hold.

> If X is second countable, metrizable or completely metrizable, then X is first countable so, by the first part of the proof, condition 2) must hold.
>
> If X is second countable or metrizable then every subspace has these same properties – so each X_α is second countable or metrizable respectively. If X is completely metrizable, then X and all the subspaces X_α are T_1. By Lemma 2.16, X_α is homeomorphic to a closed – therefore complete – subspace of X. Therefore X_α is completely metrizable.

It remains to show that if $P =$ "second countable," "metrizable," or "completely metrizable" and conditions 1) and 2) hold, then $X = \prod X_\alpha$ also has property P.

Suppose $P =$ "second countable."

For each α in the countable set B, let $\mathcal{B}_\alpha = \{O^1_\alpha, O^2_\alpha, ..., O^n_\alpha, ...\}$ be a countable base for X_α and let

$$\mathcal{B} = \{O : O \text{ is a finite intersection of sets of form } < O^n_\alpha > \text{, where } \alpha \in B \text{ and } n \in \mathbb{N}\}$$

$\mathcal{B}$ is countable and we claim $\mathcal{B}$ is a base for the product topology on X.

Suppose $x \in V = \, < V_{\alpha_1}, ..., V_{\alpha_k} >$, a basic open set in X. For each $i = 1, ..., k$, $x_{\alpha_i} \in V_{\alpha_i}$ so we can choose a basic open set in X_{α_i} such that $x_{\alpha_i} \in O^{n_i}_{\alpha_i} \subseteq V_{\alpha_i}$. Then $x \in O = \, < O^{n_1}_{\alpha_1}, O^{n_2}_{\alpha_2}, ..., O^{n_k}_{\alpha_k} > \, \subseteq V$ and $O \in \mathcal{B}$. Therefore V can be written as a union of sets from $\mathcal{B}$, so $\mathcal{B}$ is a base for X.

Suppose $P =$ "metrizable."

Since all the X_α's are T_1, condition 2) implies that all but countably many X_α's are singletons, which we can omit without changing X topologically. Therefore it is sufficient to prove that if each space $X_1, X_2, ..., X_n, ...$ is metrizable, then $X = \prod_{n=1}^{\infty} X_n$ is metrizable.

Let d_n be a metric for X_n, where without loss of generality, we can assume each $d_n \leq 1$ (*why?*). For points $x = (x_n), \ y = (y_n) \in X$, define

$$d(x, y) = \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n}$$

Then d is a metric on X (*check*!) and we claim that $\mathcal{T}_d$ is the product topology $\mathcal{T}$. Because X is a countable product of first countable spaces, X is first countable, so $\mathcal{T}$ can be described using sequences: it is sufficient to show that $(z_k) \to z$ in $(X, \mathcal{T})$ iff $(z_k) \to z$ in $(X, \mathcal{T}_d)$. But $(z_k) \to z$ in $(X, \mathcal{T})$ iff the (z_k) converges "coordinatewise." Therefore it is sufficient to show that:

$(z_k) \to z$ in $(X, \mathcal{T}_d)$ iff $\forall n \ (z_k(n)) \to z(n)$ in (X_n, d_n) or equivalently,
$\forall n \ \ d_n(z_k(n), z(n)) \to 0$

i) Suppose $d(z_k, z) \to 0$. Let $\epsilon > 0$ and consider any particular n_0. We can choose K so that $k \geq K$ implies $d(z_k, z) = \sum_{n=1}^{\infty} \frac{d_n(z_k(n), z(n))}{2^n} < \frac{\epsilon}{2^{n_0}}$. Then for $k \geq K$, $\frac{d_{n_0}(z_k(n_0), z(n_0))}{2^{n_0}} < \frac{\epsilon}{2^{n_0}}$, so $d_{n_0}(z_k(n_0), \, z(n_0)) < \epsilon$.
Therefore $d_{n_0}(z_k(n_0), \, z(n_0)) \to 0$.

ii) On the other hand, suppose $d_n(z_k(n), z(n)) \to 0$ for every n and let $\epsilon > 0$. Choose N so that $\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\epsilon}{2}$ and then choose K so that if $k \geq K$

$$\begin{aligned} &d_1(z_k(1), z(1))/2^1 < \tfrac{\epsilon}{2N} \\ &d_2(z_k(2), z(2))/2^2 < \tfrac{\epsilon}{2N} \\ &\quad \vdots \\ &d_N(z_k(N), z(N))/2^N < \tfrac{\epsilon}{2N} \ . \end{aligned}$$

Then for $k \geq K$ we have $d(z_k, z) = \sum_{n=1}^{\infty} \frac{d_n(z_k(n), z(n))}{2^n}$

$= \sum_{n=1}^{N} \frac{d_n(z_k(n), z(n))}{2^n} + \sum_{n=N+1}^{\infty} \frac{d_n(z_k(n), z(n))}{2^n} < N \cdot \frac{\epsilon}{2N} + \frac{\epsilon}{2} = \epsilon.$

Therefore $d(z_k, z) \to 0$.

Suppose $P =$ "completely metrizable."

Just as for $P =$ "metrizable," we can assume $X = \prod_{n=1}^{\infty} X_n$ and that d_n is a complete metric on X_n with $d_n \leq 1$. Using these d_n's, we define d in the same way. Then $\mathcal{T}_d$ is the product topology on X. We only need to show that (X, d) is complete.

Suppose (z_k) is a Cauchy sequence in (X, d). From the definition of d it is easy to see that $(z_k(n))$ is Cauchy in (X_n, d_n) for each n, so that $(z_k(n)) \to$ some point $a_n \in X_n$.
Let $a = (a_n) \in X$. Since $(z_k(n)) \to a_n$ for each n, we have $(z_k) \to a$ in the product topology $= \mathcal{T}_d$. Therefore (X, d) is complete. ∙

What is the correct formulation and proof of the theorem for $P =$ "pseudometrizable" ?

We might wonder why $P =$ "separable" is not included in Theorem 3.3. Since "separable" is a "countability property," we might hope that separability is preserved in countable products – although our experience Lindelöf spaces could make us hesitate. The explanation for the omission is that separability is actually better behaved for products than the other properties. Surprisingly, the product of as many as c separable spaces is separable, and the product of more than c nontrivial separable spaces can sometimes be separable. (*You should try to prove directly that a countable product of separable spaces is separable – remembering that in the product topology, "closeness depends on finitely many coordinates." If necessary, first look at finite products.*)

We begin the treatment of separability and products with a simple lemma which is merely set theory.

Lemma 3.4 Suppose $|A| \leq c$. There exists a countable collection $\mathcal{R}$ of subsets of A with the following property: given distinct $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$, there are disjoint sets $A_1, A_2, \ldots, A_n \in \mathcal{R}$ such that $\alpha_i \in A_i$ for each i.

Proof (*Think about how you would prove the theorem if $A = \mathbb{R}$. If you do that, then you'll see that the general case is just a "carry over" of that proof.*)

Since $|A| \leq c$, there is a one-to-one map $\phi : A \to \mathbb{R}$. Let $\mathcal{R} = \{\phi^{-1}[\,(a, b)\,] : a, b \in \mathbb{Q}\}$. For distinct $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$, we know that $\phi(\alpha_1), \phi(\alpha_2), \ldots, \phi(\alpha_n)$ are distinct real numbers. Then we can choose $a_i, b_i \in \mathbb{Q}$ so that $\phi(\alpha_i) \in (a_i, b_i)$ and so that the intervals (a_i, b_i) are pairwise disjoint. Then the sets $A_i = \phi^{-1}[(a_i, b_i)] \in \mathcal{R}$ are the ones we need. ∙

Theorem 3.5

1) Suppose $X = \prod_{\alpha \in A} X_\alpha \neq \emptyset$. If X is separable, then each X_α is separable.

2) If each X_α is separable and $|A| \leq c$, then $X = \prod_{\alpha \in A} X_\alpha$ is separable.

Part 2) of Theorem 3.5 is attributed (independently) to several people. In a slightly more general version, it is sometimes called the "Hewitt-Marczewski-Pondiczery Theorem." Here is an amusing sidelight, written by topologist Melvin Henriksen online in the Topology Atlas. A few words have been modified to conform with our notation:

> *Most topologists are familiar with the Hewitt-Marczewski-Pondiczery theorem. It states that if m is an infinite cardinal, then a product of 2^m topological spaces, each of which has a dense set of cardinality $\leq m$, also has a dense set with $\leq m$ points. In particular, the product of c separable spaces is separable (where c is the cardinal number of the continuum). Hewitt's proof appeared in [Bull. Amer. Math. Soc. 52 (1946), 641-643], Marczewski's proof in [Fund. Math. 34 (1947), 127-143], and Pondiczery's in [Duke Math. 11 (1944), 835-837]. A proof and a few historical remarks appear in Chapter 2 of Engelking's General Topology. The spread in the publication dates is due to dislocations caused by the Second World War; there is no doubt that these discoveries were made independently.*
>
> *Hewitt and Marczewski are well-known as contributors to general topology, but who was (or is) Pondiczery? The answer may be found in Lion Hunting & Other Mathematical Pursuits, edited by G. Alexanderson and D. Mugler, Mathematical Association of America, 1995. It is a collection of memorabilia about Ralph P. Boas Jr. (1912-1992), whose accomplishments included writing many papers in mathematical analysis as well as several books, making a lot of expository contributions to the American Mathematical Monthly, being an accomplished administrator (e.g., he was the first editor of Mathematical Reviews (MR) who set the tone for this vitally important publication, and was the chairman for the Mathematics Department at Northwestern University for many years and helped to improve its already high quality), and helping us all to see that there is a lot of humor in what we do. He wrote many humorous articles under pseudonyms, sometimes jointly with others. The most famous is "A Contribution to the Mathematical Theory of Big Game Hunting" by H. Petard that appeared in the Monthly in 1938. This book is a delight to read.*
>
> *In this book, Ralph Boas confesses that he concocted the name from Pondicheree (a place in India fought over by the Dutch, English and French), changed the spelling to make it sound Slavic, and added the initials E.S. because he contemplated writing spoofs on extra-sensory perception under the name E.S. Pondiczery. Instead, Pondiczery wrote notes in the Monthly, reviews for MR, and the paper that is the subject of this article. It is the only one reviewed in MR credited to this pseudonymous author.*
>
> *One mystery remains. Did Ralph Boas have a collaborator in writing this paper? He certainly had the talent to write it himself, but facts cannot be established by deduction alone. His son Harold (also a mathematician) does not know the answer to this question ...*

Proof 1) Let D be a countable dense set in X. For each α, $\pi_\alpha[D]$ is countable and dense in X_α (because $X_\alpha = \pi_\alpha[X] = \pi_\alpha[\text{cl}\, D] \subseteq \text{cl}\, \pi_\alpha[D]$). Therefore each X_α is separable.

2) Choose a family $\mathcal{R}$ as in Lemma 3.4 and for each α, let $D_\alpha = \{x_\alpha^1, x_\alpha^2, ..., x_\alpha^n, ...\}$ be a countable dense set in X_α. Define a countable set $\mathcal{S}$ by

$$\mathcal{S} = \{(A_1, ..., A_n,\, l_1, ..., l_n) : n \in \mathbb{N},\, l_i \in \mathbb{N},\, A_i \in \mathcal{R} \text{ with the } A_i\text{'s pairwise disjoint}\}.$$

For each α pick a point $p_\alpha \in X_\alpha$, and for each $2n$-tuple $s = (A_1, ..., A_n, l_1, ..., l_n) \in \mathcal{S}$, define a point $x_s \in X$ with coordinates

$$x_s(\alpha) = \begin{cases} x_\alpha^{l_i} & \text{if } \alpha \in A_i \\ p_\alpha & \text{if } \alpha \notin \bigcup A_i \end{cases}$$

Let $D = \{x_s : s \in \mathcal{S}\}$. D is countable and we claim that D is dense in X. To see this, consider any nonempty basic open set $U = \; < U_{\alpha_1}, ..., U_{\alpha_k} > \;:\;$ we will show that $U \cap D \neq \emptyset$.

For $U = \; < U_{\alpha_1}, ..., U_{\alpha_k} > \; \neq \emptyset$,

i) Choose disjoint sets $A_1, ..., A_k$ in $\mathcal{R}$ so that $\alpha_1 \in A_1, ..., \alpha_k \in A_k$.

ii) For each $i = 1, ..., k$, D_{α_i} is dense in X_{α_i} and we can pick a point $x_{\alpha_i}^{n_i}$ in $D_{\alpha_i} \cap U_{\alpha_i}$ $= \{x_{\alpha_i}^1, x_{\alpha_i}^2, ..., x_{\alpha_i}^n, ...\} \cap U_{\alpha_i}$.

Then, let $s = (A_1, ..., A_i, ..., A_k, n_1, ...n_i, ..., n_k) \in \mathcal{S}$. Because $\alpha_i \in A_i$, we have $x_s(\alpha_i) = x_{\alpha_i}^{n_i}$ $\in U_{\alpha_i}$. Therefore $x_s \in U \cap D$. •

Example 3.6 The rather abstract construction of a dense set D in the proof of Theorem 3.5 can be nicely illustrated with a concrete example. Consider $\mathbb{R}^\mathbb{R}$ ($= \prod_{r \in \mathbb{R}} X_r$, where each $X_r = \mathbb{R}$). Choose $\mathcal{R}$ to be the collection of all open intervals (a, b) with rational endpoints, and make a list these intervals as $A_1, ..., A_n, ...$. In each X_r, choose $D_r = \mathbb{Q} = \{q^1, q^2, ...q^n, ...\}$. (*Since all the D_r's are identical, we can omit the subscript "r" on the points; but just to stay consistent with the notation in the proof, we still use superscripts to index the q's.*) For each r, (arbitrarily) pick $p_r = 0 \in X_r$.

One example of a 6-tuple in the collection $\mathcal{S}$ is $s = (A_6, A_2, A_5, 2, 7, 4)$, where $A_1,\ A_2, A_3$ are disjoint open intervals with rational endpoints. The corresponding point $x_s \in \mathbb{R}^\mathbb{R}$ is the function $x_s : \mathbb{R} \to \mathbb{R}$ with

$$x_s(r) = \begin{cases} q^2 & \text{for } r \in A_6 \\ q^7 & \text{for } r \in A_2 \\ q^4 & \text{for } r \in A_5 \\ 0 & \text{for } r \notin A_6 \cup A_5 \cup A_2 \end{cases}$$

The dense set D consists of all step functions (such as x_s) that are 0 outside a finite union $A_{n_1} \cup A_{n_2} \cup ... \cup A_{n_k}$ of disjoint open intervals with rational endpoints and which have a constant rational value on each A_i.

Caution: In Example 2.12 we saw that the product topology on $\mathbb{R}^\mathbb{R}$ is the topology of pointwise convergence – that is, $(f_n) \to f$ in $\mathbb{R}^\mathbb{R}$ iff $(f_n(r)) \to f(r)$ for each $r \in \mathbb{R}$. But $\mathbb{R}^\mathbb{R}$ is not first countable (*why?*) so we cannot say that sequences are sufficient to describe the topology. In particular, if $f \in \mathbb{R}^\mathbb{R}$, then $f \in \text{cl}\, D$ but we cannot say that there must be sequence of step functions from D that converges pointwise to f.

Since $\mathbb{R}^\mathbb{R}$ is not first countable, $\mathbb{R}^\mathbb{R}$ is an example of a separable space that is neither second countable nor metrizable.

In contrast to the properties discussed in Theorem 3.3, an arbitrarily large product of nontrivial separable spaces can sometimes turn out to be separable, as the next example shows. However, Theorem 3.8 shows that for Hausdorff spaces, a nonempty product with more than c factors cannot be separable.

Example 3.7 For each $\alpha \in A$, let X_α be a set with $|X_\alpha| > 1$. Choose $p_\alpha \in X_\alpha$ and let $\mathcal{T}_\alpha = \{O \subseteq X_\alpha : p_\alpha \in O\} \cup \{\emptyset\}$. The singleton set $\{p_\alpha\}$ is dense in $(X_\alpha, \mathcal{T}_\alpha)$, so X_α is separable. If $p = (p_\alpha) \in X = \prod_{\alpha \in A} X_\alpha$, then singleton set $\{p\}$ is dense (*why? look at a nonempty basic open set* U.) So X is separable, and this does not depend on $|A|$. .

In this example, the X_α's are not T_1. But Exercise E7 shows that an arbitrarily large product of separable T_1-spaces can turn out to be separable.

Theorem 3.8 Suppose $X = \prod_{\alpha \in A} X_\alpha \neq \emptyset$ where each X_α is a T_2-space with more than one point. If X is separable, then $|A| \leq c$.

Proof For each α, we can pick a pair of disjoint, nonempty open sets U_α and V_α in X_α. Let D be a countable dense set in X and let $D_\alpha = \, < U_\alpha > \cap D$ for each α. If $\alpha \neq \beta \in A$, there is a point $p \in \, < U_\alpha, V_\beta > \cap D$ because D is dense. Then $p \in D_\alpha$ but $p \notin D_\beta = \, < U_\beta > \cap D$ since $x_\beta \notin U_\beta$: therefore $D_\alpha \neq D_\beta$. Therefore the map $\phi : A \to \mathcal{P}(D)$ given by $\phi(\alpha) = D_\alpha$ is one-to-one, so $|A| \leq |\mathcal{P}(D)| = 2^{\aleph_0} = c$. •

We saw in Corollary V.2.19 that a finite product of connected spaces is connected. The following theorem shows that connectedness actually behaves very nicely with respect to all products. The proof of the theorem is interesting because, unlike previous proofs about products, this proof uses the theorem about finite products to prove the general case.

Theorem 3.9 Suppose $X = \prod_{\alpha \in A} X_\alpha \neq \emptyset$. X is connected if and only if each X_α is connected.

Proof Suppose X is connected. Since $X \neq \emptyset$, we have $\pi_\alpha[X] = X_\alpha$ for each α. A continuous image of a connected space is connected (Example V.2.6), so each X_α is connected.

Conversely, suppose each X_α is connected. For each α, pick a point $p_\alpha \in X_\alpha$. For each finite set $F \subseteq A$, let $X_F = \prod_{\alpha \in F} X_\alpha \times \prod_{\alpha \in A - F} \{p_\alpha\}$. X_F is homeomorphic to the finite product $\prod_{\alpha \in F} X_\alpha$, so each X_F is connected. Let $D = \bigcup \{X_F : F \text{ is a finite subset of } A\}$. Each X_F contains the point $p = (p_\alpha)$, so Corollary V.2.10 tells us that D is connected. Unfortunately, $D \neq X$ (*except in trivial cases; why?*).

But we claim that D is dense in X. We need to show that $D \cap U \neq \emptyset$ for every nonempty basic open set in X. If $U = \, < U_{\alpha_1}, ..., U_{\alpha_n} >$, choose a point $x_{\alpha_i} \in U_{\alpha_i}$ for each $i = 1, ..., n$, and define a point $x \in X$ with coordinates

$$x(\alpha) = \begin{cases} x_{\alpha_i} & \text{if } \alpha = \alpha_i \\ p_\alpha & \text{if } \alpha \neq \alpha_1, \alpha_2, ..., \alpha_n \end{cases}$$

Let $F = \{a_1, \alpha_2, ..., \alpha_n\}$. Then $x \in X_F \cap U \subseteq D \cap U$, so $D \cap U \neq \emptyset$.

Therefore $X = \text{cl}\, D$ is connected (Corollary V.2.20). •

Question: is the analogue of Theorem 3.9 true for path connected spaces?

Just for reference, we state one more theorem here. We will not prove the theorem until Chapter IX, but we may use it in examples. (*Of course, the proof in Chapter IX will not depend on any of these examples!*)

Theorem 3.10 (Tychonoff Product Theorem) Suppose $X = \prod_{\alpha \in A} X_\alpha \neq \emptyset$. Then X is compact if and only if each X_α is compact.

One half ($\Rightarrow$) of the proof of Tychonoff's Theorem is very easy (*why?)*, and the easy proof that a finite product of compact spaces is compact was in Exercise IV.E.26.

Exercises

E10. Let $X = [0,1]^{[0,1]}$ have the product topology.

a) Prove that the set of all functions in X with finite range is dense in X. (*Here, we will call such functions step functions. In other settings, the definition of "step function" is more restrictive.*)

b) By Theorem 3.5, X is separable. Describe a countable set of step functions which is dense in X.

c) Let $A = \{x \in X : x \text{ is the characteristic function of a singleton set } \{r\}\}$. Prove that A, with the subspace topology, is discrete and not separable. Is A closed?

d) Prove that A has exactly one limit point, z, in X and that if N is a neighborhood of z, then $A - N$ is finite.

E11. "Boxes" of the form $\prod\{U_\alpha : \alpha \in A\}$, where U_α is open in X_α, are a base for the box topology on $\prod_{\alpha \in A} X_\alpha$. Throughout this problem, we assume that products have the box topology rather than the usual product topology.

a) Show that the "diagonal map" $f : \mathbb{R} \to \mathbb{R}^{\aleph_0}$ given by $f(x) = (x, x, x, ...)$ is not continuous, but that its composition with each projection map is continuous.

b) Show that $[0,1]^{\aleph_0}$ is not compact. *Hint: let $A_0 = [0,1)$ and $A_1 = (0,1]$. Consider the collection $\mathcal{U}$ of all sets of the form $A_{\epsilon_1} \times A_{\epsilon_2} \times ... \times A_{\epsilon_n} \times ...$, where $(\epsilon_1, \epsilon_2, ..., \epsilon_n, ...) \in \{0,1\}^{\aleph_0}$. By contrast, the Tychonoff Product Theorem (3.10) implies that $[0,1]^m$ (with the product topology) is compact for any cardinal m.*

c) Show that $\mathbb{R}^{\aleph_0}$ is not connected by showing that the set $A = \{x \in \mathbb{R}^{\aleph_0} : x \text{ is an unbounded sequence in } \mathbb{R}\}$ is clopen.

d) Suppose (X, d) and (X_α, d_α) $(\alpha \in A)$ are metric spaces. Prove that a function $f : X \to \prod X_\alpha$ (with the box topology) is continuous iff each coordinate function $f_\alpha = \pi_\alpha \circ f$ is continuous and each $x \in X$ has a neighborhood on which all but a finite number of the f_α's are constant.

E12. State and prove a theorem that gives a necessary and sufficient condition for a product of spaces to be path connected.

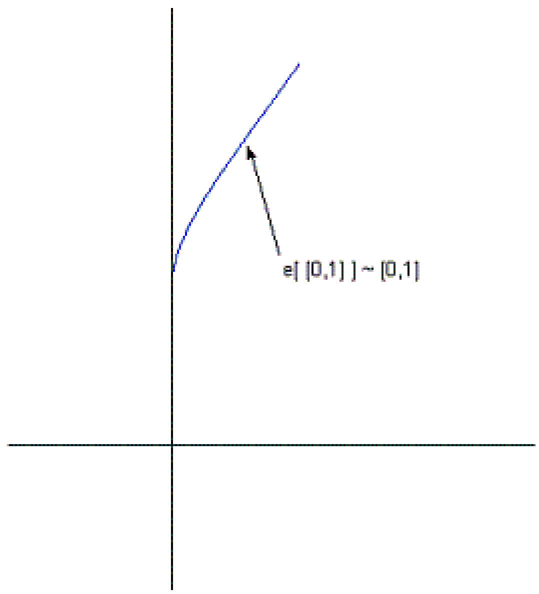
E13. Prove the following more general version of Theorem 3.8:

Suppose $X = \prod_{\alpha \in A} X_\alpha \neq \emptyset$, and that, for each $\alpha \in A$, there exist disjoint nonempty open sets U_α and V_α in X_α. If X is separable, then $|A| \leq c$.

4. Embedding Spaces in Products

If there is a homeomorphism $h : X \overset{\text{into}}{\rightarrow} Y$, then $X \simeq h[X] \subseteq Y$. We say then that X is embedded in and call h an embedding. Phrased differently, h shows that X is homeomorphic to a subspace of Y and, speaking topologically, we might say X "is" a subspace of Y. It is often possible to embed a space X in a product $Y = \prod X_\alpha$. Such embeddings will give us some nice theorems – for example, we will see that there is a separable metric space Y that contains (topologically) all other separable metric spaces – Y is a "universal" separable metric space.

To illustrate the embedding technique that we use, consider two functions $f_1 : [0,1] \rightarrow \mathbb{R}$ and $f_2 : [0,1] \rightarrow \mathbb{R}$ given by $f_1(x) = x^2$ and $f_2(x) = e^x$. Using f_1 and f_2, we can define $e : [0,1] \rightarrow \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ by using f_1 and f_2 as "coordinate functions": $e(x) = (f_1(x), f_2(x))$ $= (x^2, e^x)$. This map e is called the evaluation map defined by the set of functions $\{f_1, f_2\}$. In this example, e is an embedding – that is, e is a homeomorphism of $[0,1]$ into $\mathbb{R}^2$, so that $[0,1] \simeq \text{ran}(e) \subsetneq \mathbb{R}^2$ (*see the figure*).



An evaluation map does not always give an embedding: for example, the evaluation map $e : [0,1] \rightarrow \mathbb{R}^2$ defined by the family $\{\cos 2\pi x, \sin 2\pi x\}$ is not a homeomorphism between $[0,1]$ and $\text{ran}(e) \subseteq \mathbb{R}^2$ (*why? what is ran*(e)*?*)

We want to generalize the idea of an evaluation map e into a product and to find conditions under which e will be an embedding.

Definition 4.1 Suppose X and X_α $(\alpha \in A)$ are topological spaces and that $f_\alpha : X \rightarrow X_\alpha$ for each α. The evaluation map defined by the family $\{f_\alpha : \alpha \in A\}$ is the function $e : X \rightarrow \prod_{\alpha \in A} X_\alpha$ given by

$$e(x)(\alpha) = f_\alpha(x)$$

Thus, $e(x)$ is the point in the product $\prod X_\alpha$ whose α^{th} coordinate is $f_\alpha(x)$:. In more informal coordinate notation, $e(x) = (f_\alpha(x))$.

Exercise 4.2 Suppose $X = \prod_{\alpha \in A} X_\alpha$. For each α, there is a projection map $\pi_\alpha : X \to X_\alpha$. What is the evaluation map defined by the family $\{\pi_\alpha : \alpha \in A\}$?

Definition 4.3 Suppose X and X_α ($\alpha \in A$) are topological spaces and that $f_\alpha : X \to X_\alpha$. We say that the family $\{f_\alpha : \alpha \in A\}$ separates points if, for each pair of points $x \neq y \in X$, there exists an $\alpha \in A$ for which $f_\alpha(x) \neq f_\alpha(y)$.

Clearly, the evaluation map $e : X \to \prod_{\alpha \in A} X_\alpha$ is one-to-one $\Leftrightarrow$ for all $x \neq y \in X$, $e(x) \neq e(y)$
$\Leftrightarrow$ for all $x \neq y \in X$ there is an $\alpha \in A$ for which $e(x)(\alpha) = f_\alpha(x) \neq f_\alpha(y) = e(y)(\alpha)$
$\Leftrightarrow$ the family $\{f_\alpha : \alpha \in A\}$ separates points.

Theorem 4.4 Suppose X has the weak topology generated by the maps $f_\alpha : X \to (X_\alpha, \mathcal{T}_\alpha)$ and that the family $\{f_\alpha : \alpha \in A\}$ separates points. Then e is an embedding $-$ that is, e is a homeomorphism between X and $e[X] \subseteq \prod X_\alpha$.

Proof Since $\{f_\alpha\}$ separates points, e is one-to-one and e is continuous because each composition $\pi_\alpha \circ e = f_\alpha$ is continuous.

e preserves unions and also (since e is one-to-one) intersections. Therefore. to check that e is an open map from X to $e[X]$, it is sufficient to show that e maps subbasic open sets in X to open sets in $e[X]$. Because X has the weak topology, a subbasic open set has the form $U = f_\alpha^{-1}[V]$, where V is open in X_α. But then $e[U] = e[f_\alpha^{-1}[V]] = \pi_\alpha^{-1}[V] \cap e[X]$ is an open set in $e[X]$. $\bullet$

Note: $e[U]$ might not be open in $\prod X_\alpha$, but that is irrelevant. See the earlier example where $e(x) = (x^2, e^x)$.

The converse of Theorem 4.4 is also true: if e is an embedding, then the f_α's separate points and X has the weak topology generated by the f_α's. However, we do not need this fact and will omit the proof (which is not very hard).

Example 4.5

Let (X, d) be a separable metric space. We can assume, without loss of generality, that $d \leq 1$. X is second countable so there is a countable base $\mathcal{B} = \{U_1, ..., U_n, ...\}$ for the open sets. For each n, let $f_n(x) = d(x, X - U_n)$. Then $f_n : X \to [0,1]$ is continuous and, since $X - U_n$ is closed, we have $f_n(x) > 0$ iff $x \in U_n$. If $x \neq y \in X$, there is an n such that $x \in U_n$ and $y \notin U_n$. Then $f_n(y) = 0 \neq f_n(x)$, so f_n's separate points.

We claim that the topology $\mathcal{T}_d$ on X is the same as the weak topology $\mathcal{T}_w$ generated by the f_n's. Because the functions f_n are continuous if X has the topology $\mathcal{T}_d$, we know that $\mathcal{T}_d \supseteq \mathcal{T}_w$. To show $\mathcal{T}_d \subseteq \mathcal{T}_w$, suppose $x \in U \in \mathcal{T}_d$. For some n, we have $x \in U_n \subseteq U$ and therefore $f_n(x) = c > 0$. But $V_n = f_n^{-1}[(\frac{c}{2}, 1]]$ is a (subbasic) open set in the weak topology and $x \in V_n \subseteq U_n \subseteq U$. Therefore $U \in \mathcal{T}_w$.

By Theorem 4.4, $e : X \to [0,1]^{\aleph_0}$ is an embedding, so $X \simeq e[X] \subseteq [0,1]^{\aleph_0}$. We sometimes write this as $X \overset{\text{top}}{\subseteq} [0,1]^{\aleph_0}$. From Theorems 3.3 and 3.5, we know that $[0,1]^{\aleph_0}$ is itself a separable metrizable space $-$ and therefore all its subspaces are separable and metrizable. Putting all this together, we get that topologically, separable metrizable spaces are nothing more and nothing less than the subspaces of $[0,1]^{\aleph_0} = H$ ("the Hilbert cube").

We can view this fact with a "half-full" or "half-empty" attitude:

i) separable metric spaces must not be very complicated since topologically they are nothing more than the subspaces of a single very nice space: the "cube" $[0,1]^{\aleph_0}$

ii) separable metric spaces can get quite complicated, so the subspaces of a cube $[0,1]^{\aleph_0}$ are more complicated than we imagined.

Since $[0,1]^{\aleph_0} \simeq \prod_{n=1}^{\infty}[0,\frac{1}{n}] = H$ (the "Hilbert cube") $\subseteq \ell_2$, we can also say that topologically *the separable metrizable spaces X are* <u>*precisely*</u> *the subspaces of* ℓ_2. This is particularly amusing because of the metric d on ℓ_2 is very much like the usual metric on $\mathbb{R}^n$:

$$d(x,y) = (\sum_{n=1}^{\infty}(x_n - y_n)^2\,)^{\frac{1}{2}}$$

In some sense, this elegant "Euclidean-like" metric is adequate to describe the topology of any separable metric space. (*Note: X is topologically a subspace of H with the product topology. If we identify H with a subspace of ℓ_2, as above, how to we know that the metric topology induced on H from ℓ_2 is the same as the product topology on H?*)

We can summarize by saying that each of H and ℓ_2 is a "universal separable metric space." Notice, though, that these two "universal" spaces are not homeomorphic: one is compact and the other is not.)

Example 4.6

Suppose (X,d) is a metric space (not necessarily separable) and that $\{U_\alpha : \alpha \in A\}$ is a base for the topology $\mathcal{T}_d$, where $|A| = m$. Then an argument exactly like the one in Example 4.5 (just replace "n" everywhere with "α") shows that $X \overset{\text{top}}{\subseteq} [0,1]^m$. Therefore *every metric space, topologically, is a subspace of some sufficiently large "cube."* Of course when $m > \aleph_0$, the cube $[0,1]^m$ is <u>not</u> itself metrizable (*why?*); in general this cube will have many subspaces that are nonmetrizable. So the result is not quite as dramatic as in the separable case.

The <u>weight</u> $w(X)$ of a topological space $(X,\mathcal{T})$ is defined as min $\{|\mathcal{B}| : \mathcal{B}$ is a base for $\mathcal{T}\} + \aleph_0$.

We are assuming here that the "min" in the definition exists: see Example 5.22 in Chapter VIII. For some very simple spaces, the "min" could be finite $-$ in which case the "$+\aleph_0$" guarantees that $w(X) \geq \aleph_0$ (convenient for purely technical reasons that don't matter in these notes).

The <u>density</u> $\delta(X)$ is defined as $\min\{|D| : D$ is dense in $(X,\mathcal{T})\} + \aleph_0$. For a <u>metrizable</u> space, it is not hard to prove that $w(X) = \delta(X)$. The proof is just like our earlier proof (in Theorem III.6.5) that separability and second countability are equivalent in metrizable spaces. Therefore we have that for any metric space (X,d),

$$X \overset{\text{top}}{\subseteq} [0,1]^{w(X)} = [0,1]^{\delta(X)} \qquad (*)$$

Notice that, for a given space (X,d), the exponents in this statement are not necessarily the smallest possible. For example, (*) says that $\mathbb{R} \overset{\text{top}}{\subseteq} [0,1]^{w(\mathbb{R})} = [0,1]^{\aleph_0}$, but in fact we can do much better than the exponent $\aleph_0$: $\mathbb{R} \simeq (0,1) \subseteq [0,1] = [0,1]^1$!

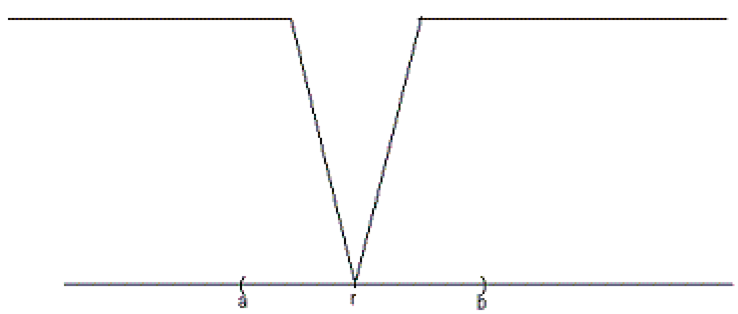
We add one additional comment, without proof: For a given infinite cardinal m, it is possible to define a <u>metrizable</u> space H_m with weight m such that <u>every</u> metric space X with weight m can be embedded in $H_m^{\aleph_0}$. In other words, $H_m^{\aleph_0}$ is a <u>metrizable</u> space which is "universal for all metric spaces of weight m." The price of metrizability, here, is that we need to replace $[0,1]$ by a more complicated space H_m.

> *Without going into all the details, you can think of H_m as a "star" with m different copies of $[0,1]$ ("rays") all placed with 0 at the center of the star. For two points x, y on the same "ray" of the star, $d(x,y) = |x-y|$; if x, y are on different rays, the distance between them is measured "via the origin" : $d(x,y) = |x| + |y|$.*

The condition in Theorem 4.4 that "X has the weak topology generated by a collection of maps $f_\alpha : X \to X_\alpha$" is not always easy to check. The following definition and theorem can sometimes help.

Definition 4.7 Suppose X and X_α $(\alpha \in A)$ are topological spaces and that $f_\alpha : X \to X_\alpha$. We say that the collection $\mathcal{F} = \{f_\alpha : \alpha \in A\}$ <u>separates points from closed sets</u> if whenever F is a closed set in X and $x \notin F$, there is an α such that $f_\alpha(x) \notin \text{cl}\, f_\alpha[F]$.

Example 4.8 Let $X = \mathbb{R}$ and $\mathcal{F} = C(\mathbb{R})$. Suppose F is a closed set in $\mathbb{R}$ and $r \notin F$. There is an open interval (a,b) for which $r \in (a,b) \subseteq \mathbb{R} - F$. Define $f \in C(\mathbb{R})$ with a graph like the one shown in the figure:



Then $0 = f(r) \notin \text{cl}\, f[F]$. Therefore $C(\mathbb{R})$ separates points and closed sets.

The same notation continues in the following lemma.

Lemma 4.9 The family $\mathcal{B} = \{f_\alpha^{-1}[V] : \alpha \in A,\ V \text{ open in } X_\alpha\}$ is a <u>base</u> for the topology in X iff

$$\begin{cases} \text{i) the } f_\alpha\text{'s are continuous, and} \\ \text{ii) } \{f_\alpha : \alpha \in A\} \text{ separates points and closed sets.} \end{cases}$$

In particular, i) + ii) imply that $\mathcal{B}$ is a subbase for the topology on X – so that X has the weak topology generated by the f_α's.

Note: the more open (closed) sets there are in X, the harder it is for a given family $\{f_\alpha : \alpha \in A\}$ to succeed in separating points and closed sets. In fact, the lemma shows that a family of continuous functions $\{f_\alpha : \alpha \in A\}$ succeeds in separating points and closed sets only if $\mathcal{T}$ is the smallest topology that makes the f_α's continuous.

Proof Suppose $\mathcal{B}$ is a base. Then the sets $f_\alpha^{-1}[V]$ in $\mathcal{B}$ are open, so the f_α's are continuous and i) holds.

To prove ii), suppose F is closed in X and $x \notin F$. For some α and some V open in X_α, we have $x \in f_\alpha^{-1}[V] \subseteq X - F$. Then $f_\alpha(x) \in V$ and $V \cap f_\alpha[F] = \emptyset$ (since $f_\alpha^{-1}[V] \cap F = \emptyset$). Therefore $f_\alpha(x) \notin \text{cl}\, f_\alpha[F]$ so ii) also holds.

Conversely, suppose i) and ii) hold. If $x \in O$ and O is open in X, we need to find a set $f_\alpha^{-1}[V] \in \mathcal{B}$ such that $x \in f_\alpha^{-1}[V] \subseteq O$. Since $x \notin F = X - O$, condition ii) gives us an α for which $f_\alpha(x) \notin \text{cl}\, f_\alpha[F]$. Then $f_\alpha(x) \in V = X_\alpha - \text{cl}\, f_\alpha[F]$, so $x \in f_\alpha^{-1}[V]$ and we claim $f_\alpha^{-1}[V] \subseteq O$:

If $w \notin O$, then $w \in F$, so $f_\alpha(w) \in f_\alpha[F] \subseteq \text{cl}\, f_\alpha[F]$.
Then $f_\alpha(w) \notin V$, so $w \notin f_\alpha^{-1}[V]$. •

Theorem 4.10 Suppose $f_\alpha : X \to X_\alpha$ is continuous for each $\alpha \in A$. If the collection $\{f_\alpha : \alpha \in A\}$

$$\begin{cases} \text{i) separates points and closed sets, and} \\ \text{ii) separates points} \end{cases}$$

then the evaluation map $e : X \to \prod X_\alpha$ is an embedding.

Proof Since the f_α's are continuous, Lemma 4.9 gives us that X has the weak topology. Then Theorem 4.4 implies that e is an embedding. •

If the space X (that we are trying to embed in a product) is a T_1-space (*as is most often the case*), then i) $\Rightarrow$ ii) in Theorem 4.10 , so we have the simpler statement given in the following corollary.

Corollary 4.11 Suppose $f_\alpha : X \to X_\alpha$ is continuous for each $\alpha \in A$. If X is a T_1-space and $\{f_\alpha : \alpha \in A\}$ separates points and closed sets, then evaluation map $e : X \to \prod X_\alpha$ is an embedding.

Proof By Theorem 4.10, it is sufficient to show that the f_α's separate points, so suppose $x \neq y \in X$. Since x is not in the closed set $\{y\}$, there is an α for which $f_\alpha(x) \notin \text{cl}\, f_\alpha[\{y\}]$. Therefore $f_\alpha(x) \neq f_\alpha(y)$, so $\{f_\alpha : \alpha \in A\}$ separates points. •

Exercises

E14. A space $(X, \mathcal{T})$ is called a T_0 space if whenever $x \neq y \in X$, then $\mathcal{N}_x \neq \mathcal{N}_y$ (equivalently, either there is an open set U containing x but not y, or vice-versa). Notice that the T_0 condition is weaker than T_1 (*see example III.2.6.4).* Clearly, a subspace of a T_0-space is T_0.

a) Prove that a nonempty product $X = \prod\{X_\alpha \colon \alpha \in A\}$ is T_0 iff each X_α is T_0.

b) Let S be "Sierpinski space" – that is, $S = \{0,1\}$ with the topology $\mathcal{T} = \{\emptyset, \{1\}, \{0,1\}\}$. Use the embedding theorems to prove that a space X is T_0 iff X is homeomorphic to a subspace of S^m for some cardinal m. *Hint* $\Rightarrow$ *: for each open set* U *in* X*, let* χ_U *be the characteristic function of* U*. Use an embedding theorem. Nearly all interesting spaces are* T_0*, and those spaces, topologically, are all just subspaces of* S^m *for some* m*.*

E15. a) Let (X, d) be a metric space. Prove that $C(X) = \{f \in \mathbb{R}^X : f \text{ is continuous}\}$ separates points and closed sets. (*Since* X *is* T_1*,* $C(X)$ *therefore also separates points*).

b) Suppose X is any T_0 topological space for which $C(X)$ separates points and closed sets. Prove that X can be embedded in a product of copies of $\mathbb{R}$.

E16. A space X satisfies the countable chain condition (CCC) if every collection of nonempty pairwise disjoint open sets must be countable. (*For example, every separable space satisfies CCC.*)

Suppose that X_α is separable for each $\alpha \in A$. Prove that $X = \prod\{X_\alpha \colon \alpha \in A\}$ satisfies CCC. (*There isn't much to prove when* $|A| \leq c$*; why?*)

Hint: Let $\{U_t : t \in T\}$ *be any such collection. We can assume all the* U_t*'s are basic open sets (why?). Prove that if* $S \subseteq T$ *and* $|S| \leq c$*, then* $|S| \leq \aleph_0$ *and hence* T *must be countable.)*

E17. There are several ways to define dimension for topological spaces. One classical method is the following inductive definition.

Define dim $\emptyset = -1$.

For $p \in X$*, we say that* X *has dimension* ≤ 0 *at* p *if there is a neighborhood base at* p *consisting of sets with* -1 *dimensional (that is, empty) frontiers. Since a set has empty frontier iff it is clopen,* X *has dimension* ≤ 0 *at* p *iff* p *has a neighborhood base consisting of clopen sets.*

We say X *has dimension* $\leq n$ *at* p *if there exists a neighborhood base at* p *in which the frontier of every basic neighborhood has dimension* $\leq n-1$.

We say X *has dimension* $\leq n$*, and write dim*$(X) \leq n$*, if* X *has dimension* $\leq n$ *at* p *for each* $p \in X$ *and that* $\boldsymbol{dim(X) = n}$ *if dim*$(X) \leq n$ *but dim*$(X) \not\leq n-1$*. We say* $\boldsymbol{dim(X) = \infty}$ *if dim*$(X) \leq n$ *is false for every* $n \in \mathbb{N}$*.*

While this definition of dim(X) *makes sense for any topological space* X*, it turns out that "dim" produces a nicely behaved dimension theory only for separable metric spaces. The dimension function "dim" is sometimes called* ***small inductive dimension*** *to distinguish it from other more general definitions of dimension. The classic discussion of small inductive dimension is in Dimension Theory (Hurewicz and Wallman).*

It is clear that "dim$(X) = n$" is a topological property. There is a theorem stating that dim$(\mathbb{R}^n) = n$, from which it follows that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ if $m \neq n$. In proving the theorem, showing dim$(\mathbb{R}^n) \leq n$ is easy; the hard part is showing that dim$(\mathbb{R}^n) \nleq n-1$.

a) Prove that $\dim(\mathbb{R}) = 1$.

b) Let C be the Cantor set. Prove that dim$(C) = 0$.

c) Suppose (X, d) is a 0-dimensional separable metric space. Prove that X is homeomorphic to a subspace of *C*. (*Hint: Show that X has a countable base of clopen sets. View C as $\{0,2\}^{\aleph_0}$ and apply the embedding theorems.*)

E18. Suppose X and Y are T_0-spaces. Part a) outlines a sufficient condition that Y can be embedded in a product of X's, i.e., that $Y \overset{\text{top}}{\subseteq} X^m$ for some cardinal m. Parts b) and c) look at some corollaries.

a) **Theorem** Let X and Y be T_0 spaces. Then Y can be topologically embedded in X^m for some cardinal m if for every closed set $F \subseteq Y$ and every point $y \notin F$, there exists a $n \in \mathbb{N}$ and a continuous function $f : Y \to X^n$ such that $f(y) \notin \text{cl}\, f[F]$.

Proof Let $T = \{t : t = (y, F),\ F$ is closed in Y and $y \notin F\}$. For each such pair $t = (y, F)$, let f_t be the function given in the hypothesis. Let $X_t = X^n$ (the space containing the range of f_t). Then clearly $\prod\{X_t : t \in T\}$ is homeomorphic to X^m for some m, so it suffices to show Y can be embedded in $\prod\{X_t : t \in T\}$. Let $h : Y \to \prod\{X_t : t \in T\}$ be defined as follows: for $y \in Y$, $h(t)$ has for its t-th coordinate $f_t(y)$, i.e., $h(y)(t) = f_t(y)$.

i) Show h is continuous.

ii) Show h is one-to-one.

iii) Show that h is a closed mapping onto its range $h[Y]$ to complete the proof that h is a homeomorphism between Y and $h[Y] \subseteq \prod\{X_t : t \in T\}$. (*Note: the converse of the theorem is also true. Both the theorem and converse are due to S. Mrowka.*)

b) Let *F* denote Sierpinski space $\{\{0,1\}, \{\emptyset,\{0\},\{0,1\}\}$. Use the theorem to show that every T_0 space *Y* can be embedded in F^m for some *m*.

c) Let *D* denote the discrete space $\{0,1\}$. Use the theorem to show that every T_1-space Y satisfying dim$(Y) = 0$ (see Exercise E14) can be embedded in D^m for some *m*.
Parts b) and c) are due to Alexandroff.

Mrowka also proved that there is no T_1-space X such that every T_1-space Y can be embedded in X^m for some m.

E19. Let $X = \{0, 1\}$ with the discrete topology, and let m be an infinite cardinal.

a) Show that X^m contains a discrete subspace of cardinality m.
b) Show that $w(X^m) = m$ (*see Example 4.6*).

E20. X is called <u>totally disconnected</u> if every connected subset A satisfies $|A| \leq 1$. Prove that a totally disconnected compact Hausdorff space is homeomorphic to a closed subspace of $\{0, 1\}^m$ for some m. (*Hint: see Lemma V.5.6*).

E21. Suppose X is a countable space that does not have a countable neighborhood base at the point $a \in X$. (*For instance, let* $a = (0, 0)$ *in the space* L*, Example III.9.8.*)

Let $A_j = \{x \in X^{\aleph_0} : x_i = a \text{ for } i > j\}$ and $A = \bigcup_{j=0}^{\infty} A_j \subseteq X^{\aleph_0}$. Prove that <u>no</u> point in the (countable) space A has a countable neighborhood base. (*Note: it is not necessary that* X *be countable. That condition simply forces* A *to be countable and makes the example more dramatic.*)

5. The Quotient Topology

Suppose that for each $\alpha \in A$ we have a map $g_\alpha : X_\alpha \to Y$, where X_α is a topological space and Y is a set. Certainly there is a topology for Y that will make all the g_α's continuous: for example, the trivial topology on Y. But what is the largest topology on Y that will do this? Let

$$\mathcal{T} = \{O \subseteq Y : \text{for all } \alpha \in A,\ g_\alpha^{-1}[O] \text{ is open in } X_\alpha\}$$

It is easy to check that $\mathcal{T}$ is a topology on Y. For each α, by definition, each set $g_\alpha^{-1}[O]$ is open in X_α, so $\mathcal{T}$ makes all the g_α's continuous. Moreover, if $B \subseteq Y$ and $B \notin \mathcal{T}$, then for at least one α, $g_\alpha^{-1}[B]$ is not open in X_α – so adding B to $\mathcal{T}$ would "destroy" the continuity of at least one g_α. Therefore $\mathcal{T}$ is the largest possible topology on Y making all the g_α's continuous.

Definition 5.1 Suppose $g_\alpha : X_\alpha \to Y$ for each $\alpha \in A$. The strong topology $\mathcal{T}$ on Y generated by the maps g_α is the largest topology on Y making all the maps g_α continuous, and $O \in \mathcal{T}$ iff $g_\alpha^{-1}[O]$ is open in X_α for every α.

The strong topology generated by a collection of maps $\{g_\alpha : \alpha \in A\}$ is "dual" to the weak topology in the sense that it involves essentially the same notation but with "all the arrows pointing in the opposite direction." For example, the following theorem states that a map f out of a space with the strong topology is continuous iff each map $f \circ g_\alpha$ is continuous; but a map f into a space with the weak topology generated by mappings f_α is continuous iff all the compositions $f_\alpha \circ f$ are continuous (*see Theorem 2.6*)

Theorem 5.2 Suppose Y has the strong topology generated by a collection of maps $\{g_\alpha : \alpha \in A\}$. If Z is a topological space and $f : Y \to Z$, then f is continuous if and only if $f \circ g_\alpha : X_\alpha \to Z$ is continuous for each $\alpha \in A$.

Proof For each $\alpha \in A$, we have $X_\alpha \xrightarrow{g_\alpha} Y \xrightarrow{f} Z$, and the g_α's are continuous since Y has the strong topology.

If f is continuous, so is each composition $f \circ g_\alpha$.

Conversely, suppose each $f \circ g_\alpha$ is continuous and that U is open in Z. We want to show that $f^{-1}[U]$ is open in Y. But Y has the strong topology, so $f^{-1}[U]$ is open in Y iff each $g_\alpha^{-1}[f^{-1}[U]]$ is open in X_α. But $g_\alpha^{-1}[f^{-1}[U]] = (f \circ g_\alpha)^{-1}[U]$ which is open because $f \circ g_\alpha$ is continuous. •

We introduced the idea of the strong topology as a parallel to the definition of weak topology. However, we are going to use the strong topology only in a special case: when there is just one map $g_\alpha = g$ and g is onto.

Definition 5.3 Suppose $(X, \mathcal{T})$ is a topological space and $g : X \to Y$ is onto. The strong topology on Y generated by g is also called the quotient topology on Y. If Y has the quotient topology from g, we say that $g : X \to Y$ is a quotient mapping and we say Y is a quotient of X. We also say the Y is a quotient space of X and sometimes as $Y = X/g$.

From the discussion of strong topologies, we know that if X is a topological space and $g : X \to Y$, then the quotient topology on Y is $\mathcal{T} = \{U : g^{-1}[U] \text{ is open in } X\}$. Thus $U \in \mathcal{T}$ if and only if $g^{-1}[U]$ is open in X: notice in this description that

"only if"	guarantees that g is continuous and
"if"	guarantees that $\mathcal{T}$ is the largest topology on Y making g continuous.

Quotients of X are used to create new spaces Y by "pasting together" ("identifying") several points of X to become a single new point. Here are two intuitive examples:

i) Begin with $X = [0, 1]$ and identify 0 with 1 (that is, "paste" 0 and 1 together to become a single point). The result is a circle, S^1. This identification is exactly what the map $g : [0, 1] \to S^1$ given by $g(x) = (\cos 2\pi x, \sin 2\pi x)$ accomplishes. It turns out (*see below*) that the usual topology on S^1 is the same as quotient topology generated by the map g. Therefore we can say that g is a quotient map and that S^1 is a quotient of $[0, 1]$

ii) If we take the space $X = S^1$ and use a mapping g to "identify" the north and south poles together, the result is a "figure-eight" space Y. The usual topology on Y (from $\mathbb{R}^2$) turns out to be the same as quotient topology generated by g (*see below*). Therefore we can say that g is a quotient mapping and the "figure-eight" is a quotient of S^1.

Suppose we are given an onto map $g : (X, \mathcal{T}) \to (Y, \mathcal{T}')$. How can we tell whether g is a quotient map – that is, how can we tell whether $\mathcal{T}'$ is the quotient topology? By definition, we must check that $U \in \mathcal{T}'$ iff $g^{-1}[U] \in \mathcal{T}$. Sometimes it is fairly straightforward to do this. But the following theorem will sometimes make things much easier.

Theorem 5.4 Suppose $g : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous and onto. If g is open (or closed), then $\mathcal{T}'$ is the quotient topology, so g is a quotient map. In particular, if X is compact and Y is Hausdorff, g is a quotient mapping.

Note: Whether $g : X \to Y$ is continuous depends, of course, on the topology $\mathcal{T}'$, but if $\mathcal{T}'$ makes g continuous, then so would any smaller topology on Y. The theorem tells us that if g is both continuous and open (or closed), then $\mathcal{T}'$ is completely determined by g: it is the largest topology that makes g continuous.

Proof Suppose $U \subseteq Y$. We must show $U \in \mathcal{T}'$ iff $g^{-1}[U] \in \mathcal{T}$. If $U \in \mathcal{T}'$, then $g^{-1}[U] \in \mathcal{T}$ because g is continuous. On the other hand, suppose $g^{-1}[U] \in \mathcal{T}$. Since g is onto and open, $g[g^{-1}[U]] = U \in \mathcal{T}'$.

For $F \subseteq Y$, $g^{-1}[Y - F] = X - g^{-1}[F]$. It follows easily that F is closed in the quotient space if and only if $g^{-1}[F]$ is closed in X. With that observation, the proof that a continuous, closed, onto map g is a quotient map is exactly parallel to the case when g is open.

If X is compact and Y is T_2, then g must be closed, so g is a quotient mapping. •

Note: Theorem 5.4 implies that the map $g(x) = (cos\,2\pi x, sin\,2\pi x)$ from $[0,1]$ to S^1 is a quotient map, but g is not open. The same formula g defines a quotient map $g : \mathbb{R} \to S^1$ which is not closed (why?). Exercise E25 gives examples of quotient maps g that are neither open nor closed.

Suppose $\sim$ is an equivalence relation on a set X. For each $x \in X$, the equivalence class of x is $[x] = \{z \in X : z \sim x\}$. The equivalence classes partition X into a collection of nonempty pairwise disjoint sets. Conversely, it is easy to see that any partition of X is the collection of equivalence classes for some equivalence relation $-$ namely, $x \sim z$ iff x and z are in the same set of the partition.

The set of equivalence classes, $Y = \{[x] : x \in X\}$, is sometimes written as $X/\sim$. There is a natural onto map $g : X \to X/\sim\ = Y$ given by $g(x) = [x]$. We can think of the elements of Y as "new points" which are created by "identifying together as one" all the members of each equivalence class in X. Conversely, whenever $g : X \to Y$ is any onto mapping, we can think of Y as the set of equivalence classes for some equivalence relation on $X -$ namely $x \sim y \iff y \in g^{-1}(x) \iff g(y) = g(x)$. If X is a topological space, we can give the set of equivalence classes Y the quotient topology.

Example 5.5 For $a,\ b \in \mathbb{Z}$, define $a \sim b$ iff $b - a$ is even. There are two equivalence classes $[0] = \{...,-4,-2,0,2,4,...\}$ and $[1] = \{...-3,-1,1,3,...\}$ so $\mathbb{Z}/\sim\ = \{[0],\ [1]\}$.

Define $g : \mathbb{Z} \to \mathbb{Z}/\sim$ by $g(a) = [a]$ and give $\mathbb{Z}/\sim$ the quotient topology. A set U is open in $\mathbb{Z}/\sim$ iff $g^{-1}[U]$ is open in $\mathbb{Z}$. But that is true for every $U \subseteq \mathbb{Z}/\sim$ because $\mathbb{Z}$ is discrete. Therefore the quotient $\mathbb{Z}/\sim$ is a two point discrete space.

Example 5.6 Let (X,d) be a pseudometric space. Define an equivalence relation $\sim$ in X by $x \sim z$ iff $d(x,z) = 0$. Let $Y = X/\sim$ and define $g : X \to Y$ by $g(x) = [x]$. Give Y the quotient topology. Then points at distance 0 in X have been "identified with each other" to become one point (an equivalence class) in Y.

For $[x],[z] \in Y$, define $d'([x],[z]) = d(x,z)$. In order to see that d' is well-defined, we need to check that the definition is independent of the representatives chosen from the equivalence classes:

If $[x'] = [x]$ and $[z'] = [z]$, then $d(x,x') = 0$ and $d(z,z') = 0$. Therefore
$d(x,z) \le d(x,x') + d(x',z') + d(z',z) = d(x',z')$, and similarly
$d(x',z') \le d(x,z)$. Thus $d(x',z') = d(x,z)$ so $d'([x'],[z']) = d'([x],[z])$.

It is easy to check that d' is a pseudometric on Y. In fact, d' is a metric: if $d'([x],[z]) = 0$, then $d(x,z) = 0$, which means that $x \sim z$ and $[x] = [z]$.

We now have two definitions for topologies on Y: the quotient topology $\mathcal{T}$ and the metric topology $\mathcal{T}_{d'}$. We claim that $\mathcal{T} = \mathcal{T}_{d'}$. To see this, first notice that

$$[y] \in B^{d'}_\epsilon([x]) \text{ iff } d'([y],[x]) < \epsilon \text{ iff } d(y,x) < \epsilon \text{ iff } y \in B^d_\epsilon(x)$$

Therefore $g^{-1}[B^{d'}_\epsilon([x])] = B^d_\epsilon(x)$ and $g\,[B^d_\epsilon(x)] = B^{d'}_\epsilon([x])$. But then we have

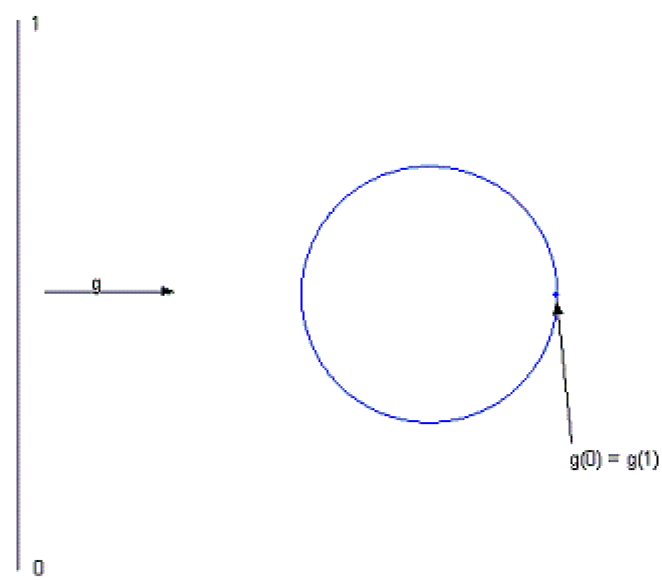
$$\begin{aligned} U \in \mathcal{T}_{d'} \quad & \text{iff } U \text{ is a union of } d'\text{-balls} \\ & \text{iff } g^{-1}[U] \text{ is a union of } d\text{-balls} \\ & \text{iff } g^{-1}[U] \text{ is open in } X \\ & \text{iff } U \in \mathcal{T}. \end{aligned}$$

The metric space (Y, d') is called the metric identification of the pseudometric space (X, d). In effect, we turn the pseudometric space into a metric space by agreeing that points in X at distance 0 are "lumped together" into a single point.

Note: In this particular example, it is easy to verify that the quotient mapping $g : X \to Y$ is open, so g would be a homeomorphism if only g were one-to-one. If the original pseudometric d is actually a metric, then g is one-to-one and a homeomorphism: the metric identification of a metric space (X, d) is itself.

Example 5.7 What does it mean if we say "identify together the endpoints of $[0, 1]$ and get a circle"? Of course, one could simply take this to be the definition of a (topological) "circle." Or, it could mean that we already know what a circle is and are claiming that a certain quotient space is homeomorphic to a circle. We take the latter point of view.

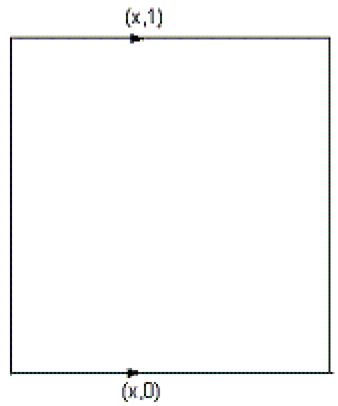
Define $g : [0, 1] \to S^1$ by $g(x) = (\cos 2\pi x, \sin 2\pi x)$. This map is onto would be one-to-one except that $g(0) = g(1)$, so g corresponds to the equivalence relation on X for which $0 \sim 1$ (and there are no other equivalences except that $x \sim x$ for every x). We can think of the equivalence classes $X/\sim$ as corresponding in a natural way to the points of S^1.



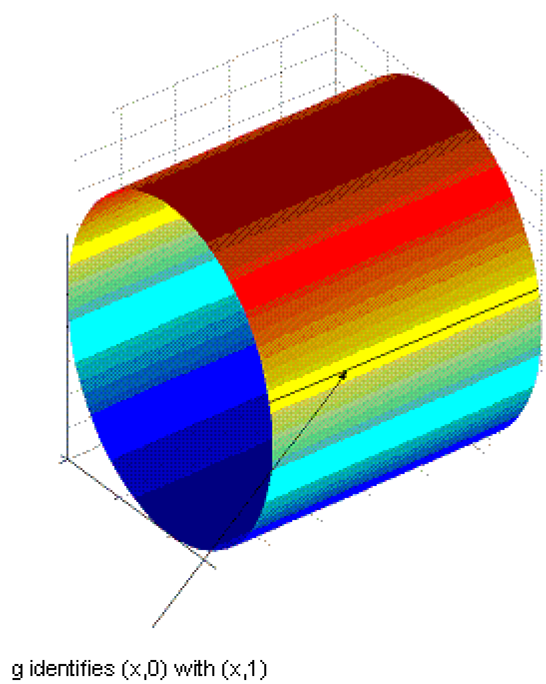
Here S^1 has its usual topology and g is continuous. Since X is compact and S^1 is Hausdorff, Theorem 5.4 gives that the usual topology on S^1 is the quotient topology and g is a quotient map.

When it "seems apparent" that the result of making certain identifications produces some familiar space Y, we need to check that the familiar topology on Y is actually the quotient topology. Example 5.7 is reassuring: if we believed, intuitively, that the result of identifying the endpoints of $[0, 1]$ should be S^1 but then found that the quotient topology on the set $X/\sim$ differed from the usual topology, we would be inclined to think that we had made the "wrong" definition for a quotient.

Example 5.8 Suppose we take a square $[0,1]^2$ and identify points on the top and bottom edges using the equivalence relation $(x,0) \sim (x,1)$. We can schematically picture this identification as

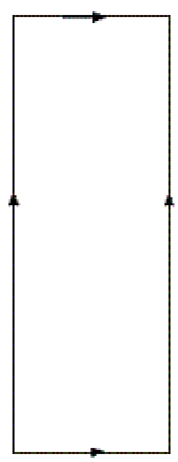


The arrows indicate that the edges are to be identified as we move along the top and bottom edges in the same direction. We have an obvious map g from $[0,1]^2$ to a cylinder in $\mathbb{R}^3$ which identifies points in just this way, and we can think of the equivalence classes as corresponding in a natural way to the points of the cylinder.

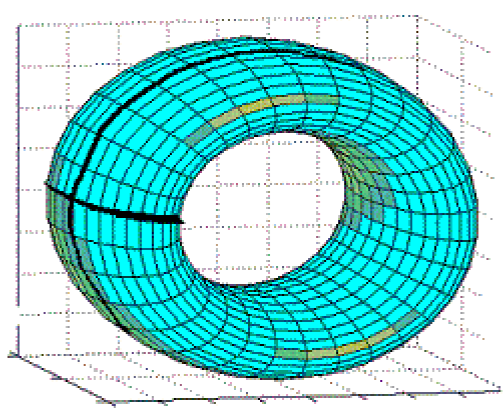


The cylinder has its usual topology from $\mathbb{R}^3$ and the map g is (clearly) continuous and onto. Again, Theorem 5.4 gives that the usual topology on the cylinder is, in fact, the quotient topology.

Example 5.9 Similarly, we can show that a torus (the "surface of a doughnut") is the result of the following identifications in $[0, 1]^2$: $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$



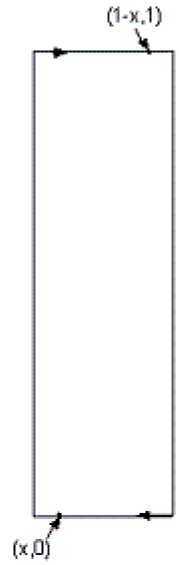
Thinking in two steps, we see that the identification of the two vertical edges produces a cylinder; the circular ends of the cylinder are then identified (in the same direction) to produce the torus.



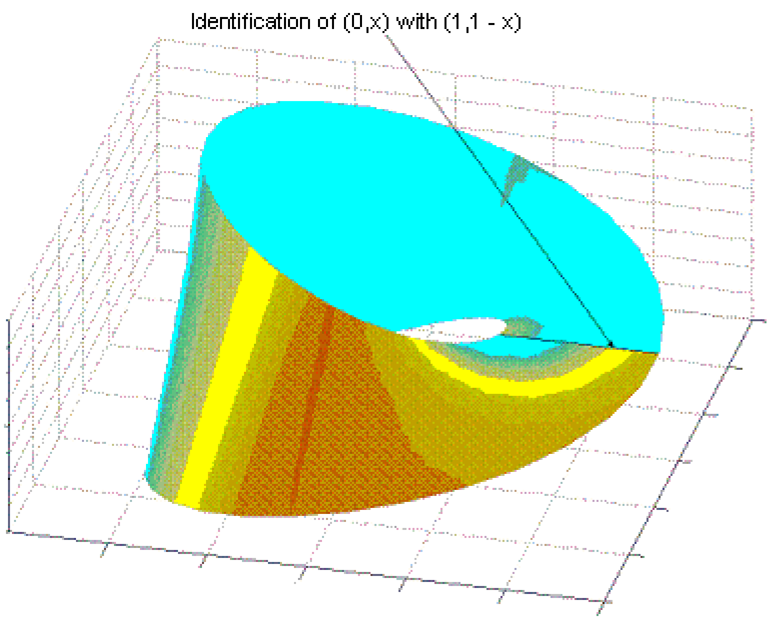
The two circles darkly shaded on the surface represent the identified edges.

We can identify the equivalence classes naturally with the points of this torus in $\mathbb{R}^3$ and just as before we see that the usual topology on the torus is in fact the quotient topology.

Example 5.10 Define an equivalence relation $\sim$ in $[0,1]^2$ by setting $(x,0) \sim (1-x,1)$. Intuitively, the idea is to identify the points on the top and bottom edges with each other as we move along the edges in opposite directions. We can picture this schematically as

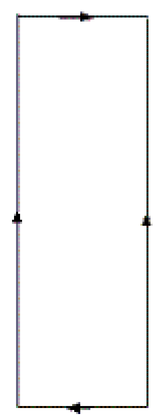


Physically, we can think of a strip of paper and glue the top and bottom edges together after making a "half-twist." The quotient space $X/\sim$ is called a Möbius strip.

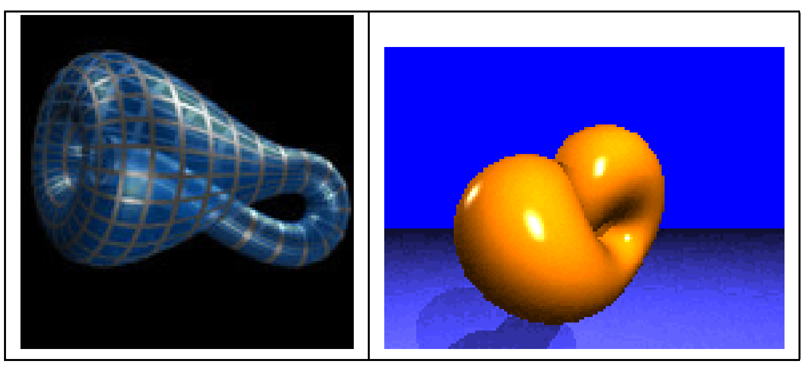


We can take the quotient $X/\sim$ as the definition of a Möbius strip, or we can consider a "real" Möbius strip M in $\mathbb{R}^3$ and define a map $g:[0,1]^2 \to M$ that accomplishes the identification we have in mind. In that case there is a natural way to associate the equivalence classes to the points of the torus in $\mathbb{R}^3$ and again Theorem 5.4 guarantees that the usual topology on the Möbius strip is the quotient topology.

Example 5.11 If we identify the vertical edges of $[0,1]^2$ (to get a cylinder) and then identify its circular ends with a half-twist (reversing orientation): $(0,y) \sim (1,y)$ and $(x,0) \sim (1-x,1)$. We get a quotient space which is called a Klein bottle.



It turns out that a Klein bottle cannot be embedded in $\mathbb{R}^3$ – the physical construction would require a "self-intersection" (that is, additional points identified) which is not allowed. A pseudo-picture looks like



In these pictures, the thin "neck" of the bottle actually intersects the main body in order to re-emerge "from the inside" – in a "real" Klein bottle (in $\mathbb{R}^4$, say), the self-intersection would not happen.

In fact, you can imagine the Klein bottle as a subset of $\mathbb{R}^4$ using color as a 4$^{\text{th}}$ dimension. To each point (x,y,z) on the "bottle" pictured above, add a 4th coordinate to get (x,y,z,r). Now color the points on the bottle in varying shades of red and let r be a number measuring the "intensity of red coloration at a point." Do the coloring in such a way that the surface "blushes" as it intersects itself – so that the points of "intersection" seen above in $\mathbb{R}^3$ will be different (in their 4$^{\text{th}}$ coordinates).

Alternately, you can think of the Klein bottle as a parametrized surface traced out by a moving point $P = (x,y,z,t)$ where x,y,z depend on time t and t is recorded as a 4$^{\text{th}}$-coordinate. A point on the surface then has coordinates of form (x, y, z, t). At "a point" where we see a self-intersection in $\mathbb{R}^3$, there are really two different points (with different time coordinates t).

Example 5.12

1) In S^1, identify antipodal points – that is, in vector notation, $P \sim -P$ for each $P \in S^1$. Convince yourself that the quotient $S^1/\sim$ is S^1.

2) Let D^2 be the unit disk $\{P \in \mathbb{R}^2 : |P| \leq 1\}$. Identify antipodal points on the boundary of D^2: that is, $P \sim -P$ if $P \in S^1 \subseteq D^2$. The quotient $D/\sim$ is called the projective plane, a space which, like the Klein bottle, cannot be embedded in $\mathbb{R}^3$.

3) For any space X, we can form the product $X \times [0,1]$ and let $(x, 1) \sim (y, 1)$ for all $x, y \in X$. The quotient $(X \times [0,1])/\sim$ is called the cone over X. (*Why?*)

4) For any space X, we can form the product $X \times [-1,1]$ and define $(x, 1) \sim (y, 1)$ and $(x, -1) \sim (y, -1)$ for all $x, y \in X$. The quotient $(X \times [-1,1])/\sim$ is called the suspension of X. (*Why?*)

There is one other very simple construction for combining topological spaces. It is often used in conjunction with quotients.

Definition 5.13 For each $\alpha \in A$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a topological space, and assume that the sets X_α are pairwise disjoint. The topological sum (or "free sum") of the X_α's is the space $(\bigcup_{\alpha\in A} X_\alpha, \mathcal{T})$ where $\mathcal{T} = \{O \subseteq \bigcup_{\alpha\in A} X_\alpha : O \cap X_\alpha$ is open in X_α for every $\alpha \in A\}$. We denote the topological sum by $\sum_{\alpha\in A} X_\alpha$. In the case $|A| = 2$, we use the simpler notation $X_1 + X_2$.

In $\sum_{\alpha\in A} X_\alpha$, each X_α is a clopen subspace. Any set open (or closed) in X_α is open (or closed) in the sum. The topological sum $\sum_{\alpha\in A} X_\alpha$ can be pictured as a union of the disjoint pieces X_α, all "far apart" from each other – so that there is no topological "interaction" between the pieces.

Example 5.14 In $\mathbb{R}^2$, let A_1 and A_2 be open disks with radius 1 and centers at $(0,0)$ and $(3,0)$. Then topological sum $A_1 + A_2$ is the same as $A_1 \cup A_2$ with subspace topology. By contrast, let B_1 be an open disk with radius 1 centered at $(0,0)$ and let B_2 be a closed disk with radius 1 centered at $(2,0)$. Then $B_1 + B_2$ is not the same as $B_1 \cup B_2$ with the subspace topology (*why?*). Are the topologies on $C_1 + C_2$ and $C_1 \cup C_2$ the same if C_1 and C_2 are separated subsets of $\mathbb{R}^2$?

Exercise 5.15 Usually it is very easy to see whether properties of the X_α's do or do not carry over to $\sum_{\alpha\in A} X_\alpha$. For example, you should convince yourself that:

1) If the X_α's are nonempty and separable, then $\sum_{\alpha\in A} X_\alpha$ is separable iff $|A| \leq \aleph_0$.

2) If the X_α's are nonempty and second countable, then $\sum_{\alpha\in A} X_\alpha$ is second countable iff $|A| \leq \aleph_0$.

3) A function $f : \sum_{\alpha \in A} X_\alpha \to Z$ is continuous iff each $f|X_\alpha$ is continuous.

4) If $d_\alpha \leq 1$ is a metric for X_α, then the topology on $\sum_{\alpha \in A} X_\alpha$ is the same as $\mathcal{T}_d$ where

$$d(x,y) = \begin{cases} d_\alpha(x,y) & \text{if } x, y \in X_\alpha \\ 1 & \text{otherwise} \end{cases}$$

so $\sum_{\alpha \in A} X_\alpha$ is metrizable if all the X_α's are metrizable. You should be able to find similar statements for other topological properties such as first countable, second countable, Lindelöf, compact, connected, path connected, completely metrizable, ...

Definition 5.16 Suppose X and Y are disjoint topological spaces and $f : A \to Y$, where $A \subseteq X$. In the sum $X + Y$, define $x \sim y$ iff $y = f(x)$. If we form $(X + Y)/\sim$, we say that we have attached X to Y with f and write this space as $X +_f Y$.

For each $p \in f[A]$, its equivalence class under $\sim$ is $\{p\} \cup f^{-1}[\{p\}]$. You may think of the function "attaching" the two spaces by repeatedly selecting a group of points in X, identifying them together, and "sewing" them all onto a single point in Y – just as you might run a needle and thread through several points in the fabric X and then through a point in Y and pull everything tight.

Example 5.17

1) Consider disjoint cylinders X and Y. Let A be the circle forming one end of X and B the circle forming one end of Y. Let $f : A \to f[A] = B \subseteq Y$ be a homeomorphism. Then f "sews together" X and Y by identifying these two circles. The result is a new cylinder.

2) Consider a sphere $S^2 \subseteq \mathbb{R}^3$. Excise from the surface S^2 two disjoint open disks D_1 and D_2 and let $A_1 \cup A_2$ be union of the two circles that bounded those disks. Let Y be a cylinder whose ends are bounded by the union of two circles, $B_1 \cup B_2$. Let f be a homeomorphism carrying the points of A_1 and A_1 clockwise onto the points of B_1 and B_2 respectively..

Then $S^2 +_f Y$ is a "sphere with a handle."

3) Consider a sphere $S^2 \subseteq \mathbb{R}^3$. Excise from the surface S^2 an open disk and let A be the circular boundary of the hole in the surface S^2. Let M be a Möbius strip and let B be the curve that bounds it. Of course, $B \simeq S^1$. Let $f : A \to B$ be a homeomorphism. The we can use f to join the spaces by "sewing" the edge of the Möbius strip to the edge of the hole in S^2. The result is a "sphere with a crosscap."

There is a very nice theorem, which we will not prove here, which uses all these ideas. It is a "classification" theorem for certain surfaces.

Definition 5.18 A Hausdorff space X is a 2-manifold if each $x \in X$ has an open neighborhood U that is homeomorphic to $\mathbb{R}^2$. Thus, a 2-manifold looks "locally" just like the Euclidean plane. A surface is a Hausdorff 2-manifold.

Theorem 5.19 Let X be a compact, connected surface. The X is homeomorphic to a sphere S^2 or to S^2 with a finite number of handles and crosscaps attached.

You can read more about this theorem and its proof in *Algebraic Topology: An Introduction* (William Massey).

Exercises

E22. a) Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ given by $(x_1, y_1) \sim (x_2, y_2)$ iff $y_1 = y_2$. Prove that $\mathbb{R}^2/\sim$ is homeomorphic to $\mathbb{R}$.

b) Find a counterexample to the following assertion: if $\sim$ is an equivalence relation on a space X and each equivalence class is homeomorphic to the same space Y, then $(X/\sim) \times Y$ is homeomorphic to X.

Why might someone conjecture that this assertion might be true? In part a), we have $X = \mathbb{R} \times \mathbb{R}$, each equivalence class is homeomorphic to $\mathbb{R}$ and $(X/\sim) \times \mathbb{R} \simeq \mathbb{R} \times \mathbb{R} \simeq X$. In this example, you "divide out" equivalence classes that all look like $\mathbb{R}$, then "multiply" by $\mathbb{R}$, and you're back where you started.

c) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x, y) = x^2 + y^2$. Then the quotient space $\mathbb{R}^2/g$ is homeomorphic to what familiar space?

d) On $\mathbb{R}^2$, define an equivalence relation $(x_1, y_1) \sim (x_2, y_2)$ iff $x_1 + y_1^2 = x_2 + y_2^2$. Prove that $\mathbb{R}^2/\sim$ is homeomorphic to some familiar space.

e) Define an equivalence relation on $\mathbb{R}$ by $x \sim y$ if an only if $x - y \in \mathbb{Z}$. What is the quotient space $\mathbb{R}/\sim$? Explain.

E23. For $x, y \in [0, 1]$, define $x \sim y$ iff $x - y$ is rational. Prove that the corresponding quotient space $[0, 1]/\sim$ is trivial.

E24. Prove that a 1-1 quotient map is a homeomorphism.

E25. a) Let $Y_1 = [0, 1]$ with its usual topology and $Y_2 = [2, 3]$ with the <u>discrete</u> topology. Define $g : Y_1 + Y_2 \to Y_1$ by letting $g(x) = \begin{cases} x & \text{if } x \in Y_1 \\ x - 2 & \text{if } x \in Y_2 \end{cases}$. Prove that g is a quotient map that is <u>neither</u> open <u>nor</u> closed.

b) Let $\mathbb{R}^2$ have the topology $\mathcal{T}$ for which a <u>subbase</u> consists of all the usual open sets together with the singleton set $\{(0, 0)\}$. Let $\mathbb{R}$ have the usual topology and let $f : \mathbb{R}^2 \to \mathbb{R}$ be the projection $f(x, y) = x$. Prove that f is a quotient map which is neither open nor closed.

E26. State and prove a theorem of the form:

"for two disjoint subsets A and B of $\mathbb{R}^2$, $A + B$ is homeomorphic to $A \cup B$ iff ... "

E27. Let $\mathbb{R}^2$ have the topology $\mathcal{T}$ for which a subbasis consists of all the usual open sets together with the singleton set $\{(0,0)\}$. Let $\mathbb{R}$ have the usual topology and define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x$. Prove that f is a quotient map which is neither open nor closed.

E28. Let $Y_1 = \mathbb{N} + (\mathbb{N} \times (0,1))$ and let $Y_2 = Y_1 + [0,1)$. Prove Y_1 is not homeomorphic to Y_2 but that each is a continuous one-to-one image of the other

E29. Show that no continuous image of $\mathbb{R}$ can be represented as a topological sum $X + Y$, where $X, Y \neq \emptyset$. How can this be result be strengthened?

E30. Suppose X_s (s $\in S$) and Y_t ($t \in T$) are pairwise disjoint spaces. Prove that $\sum_{s\in S} X_s \times \sum_{t\in T} Y_t$ is homeomorphic to $\sum_{s\in S, t\in T} (X_s \times Y_t)$.

E31. This problem outlines a proof (due to Ira Rosenholtz) that every nonempty compact metric space X is a continuous image of the Cantor set C. From Example 4.5, we know that X is homeomorphic to a subspace of $[0,1]^{\aleph_0}$.

a) Prove that the Cantor set $C \subseteq \mathbb{R}$ consists of all reals of the form $\sum_{j=0}^{\infty} \frac{a_j}{3^j}$ where each $a_j = 0$ or 2.

b) Prove that $[0,1]$ is a continuous image of C. *Hint: Define* $g(\sum_{j=0}^{\infty} \frac{a_j}{3^j}) = \sum_{j=0}^{\infty} \frac{a_j}{2^j}$.

c) Prove that the cube $[0,1]^{\aleph_0}$ is a continuous image of C. *Hint: By Corollary 2.21,* $C \simeq \{0,2\}^{\aleph_0} \simeq C^{\aleph_0}$. *Use g from part b) to define* $f : C^{\aleph_0} \to [0,1]^{\aleph_0}$ *by* $f(x_1, x_2, ..., ...) = (g(x_1), g(x_2), ..., ...)$

d) Prove that a closed set $K \subseteq C$ is a continuous image of C. *Hint: C is homeomorphic to the "middle two-thirds" set C' consisting of all reals of the form* $\sum_{j=0}^{\infty} \frac{b_j}{6^j}$. *$C'$ has the property that if* $x, y \in C'$, *then* $\frac{x+y}{2} \notin C'$. *If K' is closed in C', we can map $C' \to K$ by sending each point x to the point in K' nearest to x.)*

e) Prove that every nonempty compact metric space X is a continuous image of C.

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Chapter VI Review

Explain why each statement is true, or provide a counterexample.

1. $(0,1)^{\aleph_0}$ is open in $[0,1]^{\aleph_0}$.

2. Suppose F is a closed set in $[0,1] \times \mathbb{R}$. Then $\pi_2[F]$ is a closed set in $\mathbb{R}$.

3. $\mathbb{N}^{\aleph_0}$ is discrete.

4. If C is the Cantor set, then there is a complete metric on $C^{\aleph_0}$ which produces the product topology.

5. Let $\mathbb{R}^{\mathbb{R}}$ have the box topology. A sequence $(f_n) \to f \in \mathbb{R}^{\mathbb{R}}$ iff $(f_n) \to f$ uniformly.

6. Let $f_n : [0,1] \to [0,1]$ be given by $f_n(x) = x^n$. The sequence (f_n) has a limit in $[0,1]^{[0,1]}$.

7. Let $g \in \mathbb{R}^{\mathbb{R}}$ be defined by $g(x) = x^2$ for all $x \in \mathbb{R}$. Give an example of a sequence (f_n) of distinct functions in $\mathbb{R}^{\mathbb{R}}$ that converges to g.

8. Let C be the Cantor set. Then C is homeomorphic to the topological sum $C + C$.

9. The projection maps π_x and π_y from $\mathbb{R}^2 \to \mathbb{R}$ separate points from closed sets.

10. The letter N is a quotient of the letter M .

11. Suppose $\sim$ is an equivalence relation on X and that for $x \in X$, $[\,x\,]$ represents its equivalence class. If x is a cut point of X, then $[\,x\,]$ is a cut point of the quotient space $X/\sim$.

12. If g: $X \to Y$ is a quotient map and Y is compact T_2, then Y is compact T_2.

13. Suppose A is infinite and that in each space X_α $(\alpha \in A)$ there is a nonempty proper open subset O_α. Then $\prod O_\alpha$ is not a <u>basic</u> open set in the product topology on $X = \prod X_\alpha$. Moreover, $\prod O_\alpha$ cannot even be open in the product.

14. If $X = \bigcup_{n=1}^{\infty} A_n$, where the A_n's are disjoint clopen sets in X, then $X \cong \sum_{n=1}^{\infty} A_n$ (= the topological sum of the A_n's).

15. Let $X_n = \{0,1\}$ and $Y_n = \mathbb{N}$ with their usual topologies. Then $\sum\limits_{n=1}^{\infty} X_n$ is homeomorphic to $\sum\limits_{n=1}^{\infty} Y_n$.

16. Let $X_n = \{0,1\}$ and $Y_n = \mathbb{N}$ with their usual topologies. Then $\prod_{n=1}^{\infty} X_n$ is homeomorphic to $\prod_{n=1}^{\infty} Y_n$.

17. Let $A = \{(x,y,z) \in \mathbb{R}^3\colon\ x - y^2 - 2yz - z^2 > |\sin(xyz)|\,\}$, and let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y,z) = x + 2$. Then $f\,[A]$ is open but not closed in $\mathbb{R}$.

18. Suppose A_n is a connected subset of $X_n (\neq \emptyset)$ and that $\prod_{n=1}^{\infty} A_n$ is dense in $\prod_{n=1}^{\infty} X_n$. Then each X_n is connected.

19. In $\mathbb{R}^{\mathbb{R}}$, every neighborhood of the function sin contains a step function (that is, a function with finite range).

20. Let X be an uncountable set with the cocountable topology. Then $\{(x, x) : x \in X\}$ is a closed subset of the product $X \times X$..

21. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$, and let m be any cardinal number. A^k is a closed set in $\mathbb{R}^k$.

22. If C is the Cantor set, then $C \times C \times C \times C$ is homeomorphic to $C \times C$.

33. $S^1 \times S^1$ is homeomorphic to the "infinity symbol": ∞

34. 31. Let P be the set of all real polynomials in one variable, with domain restricted to $[0, 1]$, for which ran$(P) \subseteq [0, 1]$. Then P is dense in $[0, 1]^{[0,1]}$.

35. Every metric space is a quotient of a pseudometric space.

37. A separable metric space with a basis of clopen sets is homeomorphic to a subspace of the Cantor set.

38. Let $\prod_{\alpha \in A} X_\alpha$ be a nonempty product space. Then each factor X_α is a quotient of $\prod_{\alpha \in A} X_\alpha$.

39. Suppose X does not have the trivial topology. Then X^{2^c} cannot be separable.

40. Every countable space X is a quotient of $\mathbb{N}$.

41. $\mathbb{N} \times \mathbb{N}$ is homeomorphic to the sum of $\aleph_0$ disjoint copies of $\mathbb{N}$.

42. Suppose $x = (x_\alpha) \in \text{int } A$, where $A \subseteq \prod X_\alpha$. Then for every α, $x_\alpha \in \text{int } \pi_\alpha[A]$.

43. The unit circle, S^1, is homeomorphic to a product $\prod_{\alpha \in A} X_\alpha$, where each $X_\alpha \subseteq [0, 1]$ (i.e., S^1 can be "factored" into a product of subspaces of $[0, 1]$).

44. $\mathbb{N}^{\aleph_0}$ is homeomorphic to $\mathbb{R}$.