Chapter X
Compactifications

1. Basic Definitions and Examples

Definition 1.1 Suppose $h : X \to Y$ is a homeomorphism of X into Y, where Y is a compact T_2 space. If $h[X]$ is dense in Y, then the pair (Y, h) is called a compactification of X.

By definition, only Hausdorff spaces X can (possibly) have a compactification.

If we are just working with a single compactification Y of X, then we can usually just assume that $X \subseteq Y$ and that h is the identity map $-$ so that the compactification is just a compact Hausdorff space that contains X as a dense subspace. In fact, if $X \supseteq Y$, we will always assume that h is the identity map unless something else is stated. We made similar assumptions in discussing of the completion of a metric space (X, d) in Chapter IV.

However, we will sometimes want to compare different compactifications of X (in a sense to be discussed later) and then we may need to know how X is embedded in Y. We will see that different dense embeddings h of X into the same space Y can produce "nonequivalent" compactifications. Therefore, strictly speaking, a "compactification of X" is the pair (Y, h).

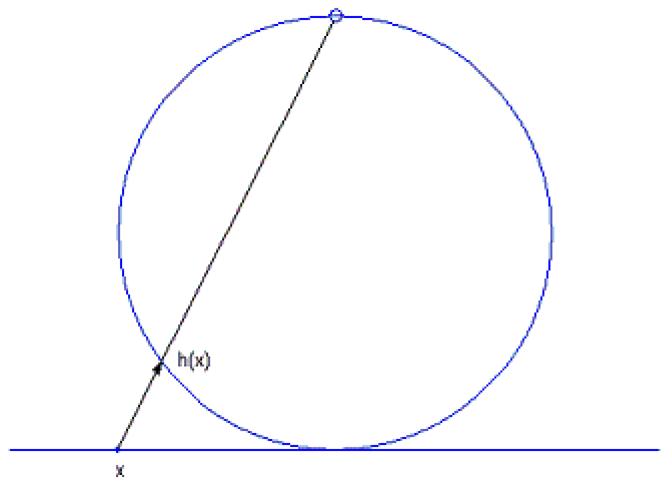
If, in Definition 1.1, X is already a compact Hausdorff space, then $h[X]$ is closed and dense in Y and therefore $h[X] = Y$. Therefore, topologically, the only possible compactification of X is X itself.

The next theorem restates exactly which spaces have compactifications.

Theorem 1.2 A space X has a compactification iff it is a Tychonoff space.

Proof See the remarks following Corollary IX.6.3. $\bullet$

Example 1.3 The circle S^1 can be viewed as a compactification of the real line, $\mathbb{R}$. Let h be the "inverse projection" pictured below: here $h[\mathbb{R}] = S^1 - \{\text{North Pole}\}$. We can think of $h[\mathbb{R}]$ as a "bent" topological copy of $\mathbb{R}$, and the compactification is created by "tying together" the two ends of $\mathbb{R}$ by adding one new "point at infinity" (the North Pole).



Since $|S^1 - h[\mathbb{R}]| = 1$, (S^1, h) is called a one-point compactification of $\mathbb{R}$. (*We will see in Example 4.2 that we can call* S^1 *the one-point compactification of* $\mathbb{R}$.)

Example 1.4

1) $[-1,1]$ is a compact Hausdorff space containing $(-1,1)$ are a dense subspace, so $[-1,1]$ is a "two-point" compactification of $(-1,1)$ (with embedding $h = i$).

2) If $h : \mathbb{R} \to (-1,1)$ is a homeomorphism, then $i \circ h : \mathbb{R} \to [-1,1]$ gives a "two-point" compactification of $\mathbb{R}$. *It is true (but not so easy to prove) that there is no* n*-point compactification of* $\mathbb{R}$ *for* $2 < n < \omega_0$.

Example 1.5 Suppose $Y = X \cup \{p\}$ is a one-point compactification of X. If O is an open set containing p in Y, then $K = Y - O \subseteq X$ and K is compact. Therefore the open sets containing p are the complements of compact subsets of X. (*Look at open neighborhoods of the North Pole* p *in the one-point compactification* S^1 *of* $\mathbb{R}$*; a base for the open neighborhoods of* p *consists of complements of the closed (compact!) arcs that do not contain the North Pole.*)

Suppose $x \in U$, where U is open in X. Because Y is Hausdorff, we can find disjoint open sets V and W in Y with $x \in V$ and $p \in W$. Since $p \notin V$, we have that $V \subseteq X$ and therefore $V \cap X = V$ is also open in X. Since $x \in U \cap V$, we can use the regularity of X to choose an open set G in X for which $x \in G \subseteq \text{cl}_X G \subseteq U \cap V \subseteq U$. But $G \subseteq V \subseteq Y - W$ (a closed set in Y), so $\text{cl}_Y G \subseteq Y - W \subseteq X$. Therefore $\text{cl}_X G = X \cap \text{cl}_Y G = \text{cl}_Y G$, so $\text{cl}_X G$ is also closed in Y. So $\text{cl}_X G$ is a compact neighborhood of x inside U. This shows that each point $x \in X$ has a neighborhood base in X consisting of compact neighborhoods.

The property in the last sentence is important enough to deserve a name: such spaces are called locally compact.

2. Local Compactness

Definition 2.1 A Hausdorff space X is called locally compact if each point $x \in X$ has a neighborhood base consisting of compact neighborhoods .

Example 2.2

1) A discrete space is locally compact.

2) $\mathbb{R}^n$ is locally compact: at each point x, the collection of closed balls centered at x is a base of compact neighborhoods. On the other hand, neither $\mathbb{Q}$ nor $\mathbb{P}$ is locally compact. (*Why?*)

3) If X is a compact Hausdorff space, then X is regular so there is a base of closed neighborhoods at each point – and each of these neighborhoods is compact. Therefore X is locally compact.

4) Each ordinal space $[0, \alpha)$ is locally compact. The space $[0, \alpha]$ is a (one-point) compactification of $[0, \alpha)$ iff α is a limit ordinal.

5) Example 1.5 shows that if a space X has a one-point compactification, it must be locally compact (and, of course, noncompact and Hausdorff). Therefore neither $\mathbb{Q}$ nor $\mathbb{P}$ has a one-point compactification. The following theorem characterizes the spaces with one-point compactifications.

Theorem 2.3 A space X has a one-point compactification iff X is a noncompact, locally compact Hausdorff space. (*The one-point compactification of X for which the embedding h is the identity is denoted X^*.*)

Proof Because of Example 1.5, we only need to show that a noncompact, locally compact Hausdorff space X has a one-point compactification. Choose a point $p \notin X$ and let $X^* = X \cup \{p\}$. Put a topology on X^* by letting each point $x \in X$ have its original neighborhood base of compact neighborhoods, and by defining basic neighborhoods of p be the complements of compact subsets of X :

$$\mathcal{B}_p = \{N \subseteq X^* : \ p \in N \text{ and } X^* - N \text{ is compact}\}.$$

(*Verify that the conditions of the Neighborhood Base Theorem III.5.2 are satisfied.*)

If $\mathcal{U}$ is an open cover of X^* and $p \in U \in \mathcal{U}$, then there exists an $N \in \mathcal{B}_p$ with $p \in N \subseteq U$. Since $X^* - N$ is compact, we can choose $U_1, ..., U_n \in \mathcal{U}$ covering $X^* - N$. Then $\{U, U_1, ..., U_n\}$ is a finite subcover of X^* from $\mathcal{U}$. Therefore X^* is compact.

X^* is Hausdorff. If $a \neq b \in X$, then a and b can be separated by disjoint open sets in X and these sets are still open in X^*. Furthermore, if K is a compact neighborhood of a in X, then K and $(X^* - K)$ are disjoint neighborhoods of a and p in X^*.

Finally, notice that $\{p\}$ is not open in X^* – or else $\{p\} \in B_p$ and then $X^* - \{p\} = X$ would be compact. Therefore every open set containing p intersects X, so X is dense in X^*.

Therefore X^* is a one-point compactification of X. $\bullet$

What happens if the construction for X^* in the preceding proof is carried out starting with a space X which is already compact? What happens if X is not locally compact? What happens if X is not Hausdorff ?

Corollary 2.4 A locally compact Hausdorff space X is Tychonoff.

Proof X is either compact or X has a one-point compactification X^*. Either way, X is a subspace of a compact T_2 space which (by Theorem VII.5.9) is Tychonoff. Therefore X is Tychonoff. $\bullet$

The following theorem about locally compact spaces is often useful.

Theorem 2.5 Suppose $A \subseteq X$, where X is Hausdorff.

a) If X is locally compact and $A = F \cap G$ where F is closed and A is open in X, then A is locally compact. In particular, an open (or, a closed) subset of a locally compact space X is locally compact.

b) If A is a locally compact and X is Hausdorff, then A is open in $\mathrm{cl}_X A$.

c) If A is a locally compact subspace of a Hausdorff space X, then $A = F \cap G$ where F is closed and A is open in X.

Proof a) It is easy to check that if F is closed and G is open in a locally compact space X, then F and G are locally compact. It then follows easily that $F \cap G$ is also locally compact. (*Note: Part a) does not require that X be Hausdorff.*)

b) Let $a \in A$ and let K be a compact neighborhood of a in A. Then $a \in \mathrm{int}_A K = U$. Since A is Hausdorff, K is closed and therefore $U \subseteq \mathrm{cl}_A U \subseteq K$, so $\mathrm{cl}_A U$ is compact.

Because U is open in A, there is an open set V in X with $A \cap V = U$ and we have:

$$\mathrm{cl}_X(A \cap V) \cap A = (\mathrm{cl}_X U) \cap A = \mathrm{cl}_A U \subseteq A$$

so $(\mathrm{cl}_X(A \cap V)) \cap A$ is compact and therefore closed in X (since X is Hausdorff).

Since $A \cap V \subseteq (\mathrm{cl}_X(A \cap V)) \cap A$, we have $\mathrm{cl}_X(A \cap V) \subseteq (\mathrm{cl}_X U) \cap A = \mathrm{cl}_A U \subseteq A$.

Moreover, since V is open, then $V \cap \mathrm{cl}_X A \subseteq \mathrm{cl}_X(V \cap A)$ (*this is true in <u>any</u> space X: why?*).

So $W = V \cap \mathrm{cl}_X A \subseteq \mathrm{cl}_X(A \cap V) \subseteq (\mathrm{cl}_X U) \cap A = \mathrm{cl}_A U \subseteq A$.

Then $a \in W \subseteq A$ and W is open in $\mathrm{cl}_X A$ so $a \in \mathrm{int}\,\mathrm{cl}_X A$. Therefore A is open in $\mathrm{cl}_X A$.

c) Since A is locally compact, part b) gives that A is open in $\mathrm{cl}_X A$, so $A = \mathrm{cl}_X A \cap G$ for some open set G in X. Let $F = \mathrm{cl}_X A$. $\bullet$

Corollary 2.6 A dense locally compact subspace of a Hausdorff space X is open in X.

Proof This follows immediately from part b) of the theorem $\bullet$

Corollary 2.7 If X is a locally compact, noncompact Hausdorff space, then X is open in <u>any</u> compactification Y that contains X.

Proof This follows immediately from Corollary 2.6. $\bullet$

Corollary 2.8 A locally compact metric space (X, d) is completely metrizable.

Proof Let $(\widetilde{X}, \widetilde{d})$ be the completion of (X, d). X is locally compact and dense in $\widetilde{X}$ so X is open in $\widetilde{X}$. Therefore X is a G_δ-set in $\widetilde{X}$ so it follows from Theorem IV.7.5 that X is completely metrizable. $\bullet$

Theorem 2.9 Suppose $X = \prod_{\alpha \in A} X_\alpha \neq \emptyset$ is Hausdorff. Then X is locally compact iff

i) each X_α is locally compact
ii) X_α is compact for all but at most finitely many $\alpha \in A$.

Proof Assume X is locally compact. Suppose U_α is open in X_α and $x_\alpha \in U_\alpha$. Pick a point $z \in \pi_\alpha^{-1}[U_\alpha]$ with $z_\alpha = x_\alpha$. Then z has a compact neighborhood K in X for which $z \in K \subseteq \pi_\alpha^{-1}[U_\alpha]$. Since π_α is an open continuous map, $\pi_\alpha[K]$ is a compact neighborhood of x_α with $x_\alpha \in \pi_\alpha[K] \subseteq U_\alpha$. Therefore X_α is locally compact, so i) is true.

To prove ii), pick a point $x \in X$ and let K be a compact neighborhood of x. Then $x \in U = \, < U_{\alpha_1}, ..., U_{\alpha_n} > \, \subseteq K$ for some basic open set U. If $\alpha \neq \alpha_1$,..., α_n, we have $\pi_\alpha[K] \supseteq \pi_\alpha[U] = X_\alpha$. Therefore X_α is compact if $\alpha \neq \alpha_1$,..., α_n.

Conversely, assume i) and ii) hold. If $x \in U \subseteq X$, where U is open, then we can choose a basic open set $V = \, < V_{\alpha_1}, ..., V_{\alpha_n} >$ so that $x \in V \subseteq U$. Without loss of generality, we can assume that X_α is compact for $\alpha \neq \alpha_1$,..., α_n (*why?*). For each i we can choose a compact neighborhood K_{α_i} of x_{α_i} so that $x_{\alpha_i} \in K_{\alpha_i} \subseteq V_{\alpha_i} \subseteq X_{\alpha_i}$. Then $< K_{\alpha_1}, ..., K_{\alpha_n} >$ $= K_{\alpha_1} \times ... \times K_{\alpha_n} \times \prod X_{\alpha \neq \alpha_1, ... \alpha_n} X_\alpha$ is a compact neighborhood of x and $x \in \, < K_{\alpha_1}, ..., K_{\alpha_n} > \, \subseteq \, < V_{\alpha_1}, ..., V_{\alpha_n} > \, \subseteq U$. So X is locally compact. $\bullet$

3. The Size of Compactifications

Suppose X is a Tychonoff space, that $X \subseteq Y$, and that Y is a compactification of X. How large can $|Y - X|$ be? In all the specific example so far, we have had $|Y - X| = 1$ or $|Y - X| = 2$.

Example 3.1 This example illustrates a compactification of a discrete space created by adding c points.

Let $I_0 = \{(x, 0) : x \in [0, 1]\}$ and $I_1 = \{(x, 1) : x \in [0, 1]\}$, two disjoint "copies" of $[0, 1]$. Let Define a topology on $Y = I_0 \cup I_1$ by using the following neighborhood bases:

i) points in I_1 are isolated: for $p \in I_1,$ a neighborhood base at p is $\mathcal{B}_p = \{\{p\}\}$

ii) if $p = (x, 0) \in I_0$: a basic neighborhood of p is any set of form $V \cup \{(z, 1) : (z, 0) \in V, z \neq x\},$ where V is an open neighborhood of p in $[0, 1]$

(*Check that the conditions in the Neighborhood Base Theorem III.5.2 are satisfied.*)

Y is called the "double" of the space $[0, 1] = I_0$.

Clearly, Y is Hausdorff, and we claim that Y is compact. It is sufficient to check that any covering $\mathcal{U}$ of Y by basic open neighborhoods has a finite subcover.

Let $\mathcal{W} = \{W \in \mathcal{U} : W \cap I_0 \neq \emptyset\}$. $\mathcal{W}$ covers I_0 and each $W \in \mathcal{W}$ has form $V \cup \{(z, 1) : (z, 0) \in V, z \neq x\}$, where V is open in I_0. Clearly, the open "V-parts" of the sets in $\mathcal{W}$ cover the compact space I_0, so we finitely many $W_1, ..., W_n \in \mathcal{W}$ cover I_0. These sets also cover I_1, <u>except</u> for possibly finitely many points $p_1, ..., p_k \in I_1$. For each such point p_i choose a set $U_i \in \mathcal{U}$ containing p_i. Then $\{W_1, ..., W_n, U_1, ..., U_k\}$ is a finite subcover from $\mathcal{U}$..

Every neighborhood of a point in I_0 intersects I_1, so $\operatorname{cl} I_1 = Y$. Therefore Y is a compactification of the discrete space I_1 and $|Y - I_1| = c$.

Since I_1 is locally compact, I_1 also has another quite different compactification I_1^* for which $|I_1^* - I_1| = 1$. In fact, it is true (depending on X) that can be many different compactifications Y, each with a different size for $|Y - X|$.

But, for a given space X and a compactification Y, there <u>is</u> an upper bound for how large $|Y - X|$ can be. We can find it using the following two lemmas.

Recall that the weight $w(Y)$ of a space $(Y, \mathcal{T})$ is defined by $w(Y) = \aleph_0 + \min\{|\mathcal{B}| : \mathcal{B}$ is a base for $\mathcal{T}\}$. (*Example VI.4.6*)

Lemma 3.2 If Y is a T_0 space, then $|Y| \leq 2^{w(Y)}$.

Proof Let $\mathcal{B}$ be <u>any</u> base for Y, and for each point $y \in Y$, let $\mathcal{B}_y = \{U \in \mathcal{B} : y \in U\}$. Since Y is T_0, we have $\mathcal{B}_{y'} \neq \mathcal{B}_y$ if $y' \neq y$. Therefore the map $y \to \mathcal{B}_y \subseteq \mathcal{B}$ is one-to-one, so

$|Y| \leq |\mathcal{P}(\mathcal{B})| = 2^{|\mathcal{B}|}$. In particular, if we pick $\mathcal{B}$ to be a base with the least possible cardinality, minimal cardinality, then $|Y| \leq 2^{|\mathcal{B}|} \leq 2^{w(Y)}$. •

Lemma 3.3 Suppose Y is an infinite T_3 space and that X is a dense subspace of Y. Then $w(Y) \leq 2^{|X|} \leq 2^{|Y|}$.

Proof A T_3 space with a finite base must be finite, so every base for Y must be infinite. Let $\mathcal{B} = \{U_\alpha : \alpha \in A\}$ be a base for Y. Each U_α is open so we have i) $U_\alpha \subseteq \operatorname{int} \operatorname{cl} U_\alpha \subseteq \operatorname{cl} U_\alpha$, and ii) because X is dense in Y, $\operatorname{cl} U_\alpha = \operatorname{cl}(U_\alpha \cap X)$ (*see Lemma IV.6.4*).

For each α, define $V_\alpha = \operatorname{int} \operatorname{cl}(U_\alpha \cap X)$, so that $U_\alpha \subseteq \operatorname{int} \operatorname{cl} U_\alpha = \operatorname{int} \operatorname{cl}(U_\alpha \cap X) = V_\alpha$.

Then $\mathcal{B}' = \{V_\alpha : \alpha \in A\}$ is also a base for Y: to see this, suppose $y \in O \subseteq Y$ where O is open. By regularity, there is a U_α such that $y \in U_\alpha \subseteq \operatorname{int} \operatorname{cl} U_\alpha = V_\alpha \subseteq \operatorname{cl} U_\alpha \subseteq O$.

Since each $U_\alpha \cap X \subseteq X$, there are no more distinct V_α's than there are subsets of X, that is $|\mathcal{B}'| \leq |\mathcal{P}(X)|$. Since $\mathcal{B}'$ must be infinite, we have $w(Y) \leq |\mathcal{B}'| \leq |\mathcal{P}(X)| = 2^{|X|} \leq 2^{|Y|}$. •

Theorem 3.4 If Y is a compactification of X and D is dense in X, then $|Y| \leq 2^{2^{|D|}}$.

Proof Y is Tychonoff. If Y is finite, then $D = X = Y$ so $|Y| \leq 2^{2^{|Y|}} = 2^{2^{|D|}}$. Therefore we can assume Y is infinite. Since D is dense in Y, $w(Y) \leq 2^{|D|}$ (by Lemma 3.3), and therefore so $|Y| \leq 2^{w(Y)} \leq 2^{2^{|D|}}$ (by Lemma 3.2) •

Example 3.5 An upper bound on the size of a compactification of $\mathbb{N}$ is $2^{2^{\aleph_0}} = 2^c$. More generally, a compactification of any separable Tychonoff space – such as $\mathbb{N}, \mathbb{Q}, \mathbb{P}$ or $\mathbb{R}$ – can have no more than 2^c points.

We will see in Section 6 that Theorem 3.4 is "best possible" upper bound. For example, there actually exists a compactification of $\mathbb{N}$, called $\beta\mathbb{N}$, with cardinality $2^{2^{\aleph_0}} = 2^c$! (It is difficult to imagine how the "tiny" discrete set $\mathbb{N}$ can be dense in a such a large compactification $\beta\mathbb{N}$.

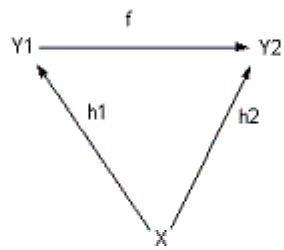
> *Assume such a compactification $\beta\mathbb{N}$ exists. Since $\mathbb{N}$ is dense, each point σ in $\beta\mathbb{N} - \mathbb{N}$ is the limit of a net in $\mathbb{N}$, and this net has a universal subnet which converges to σ.*
>
> *Since $\beta\mathbb{N}$ is Hausdorff, a universal net in $\mathbb{N}$ has at most one limit in $\beta\mathbb{N} - \mathbb{N}$, so there are at least as many universal nets in $\mathbb{N}$ as there are points in $\beta\mathbb{N} - \mathbb{N}$, namely 2^c. None of these universal nets can be trivial (that is, eventually constant). Therefore each of these universal nets is associated with a free (= nontrivial) ultrafilter in $\mathbb{N}$. So there must be 2^c free ultrafilters in $\mathbb{N}$.*

4. Comparing Compactifications

We want to compare compactifications of a Tychonoff space X. We begin by defining an equivalence relation $\simeq$ between compactifications of X. Then we define a relation $\geq$. It will turn out that $\geq$ can also be used to compare equivalence classes of compactifications of X. When applied in a set equivalence classes of compactifications of X, $\geq$ will turn out to be a partial ordering.

The definition of $\simeq$ requires that we use the formal definition of a compactification as a pair.

Definition 4.1 Two compactifications (Y_1, h_1) and (Y_2, h_2) of X are called <u>equivalent</u>, written $(Y_1, h_1) \simeq (Y_2, h_2)$, if there is a homeomorphism f of Y_1 onto Y_2 such that $f \circ h_1 = h_2$.



In the <u>special case</u> where $X \subseteq Y_1$, $X \subseteq Y_2$, and $h_1 = h_2 =$ the identity map on X, then the condition $f = f \circ h_1 = h_2$ simply states that $f(x) = x$ for $x \in X$ $-$ that is, points in X are fixed under the homeomorphism f.

It is obvious that $(Y_1, h_1) \simeq (Y_1, h_1)$ and that $\simeq$ is a transitive relation among compactifications of X. Also, if $(Y_1, h_1) \simeq (Y_2, h_2)$, then $f^{-1} : Y_2 \to Y_1$ is a homeomorphism and $f^{-1} \circ h_2 = f^{-1}(f \circ h_1) = h_1$ so that $(Y_2, h_2) \simeq (Y_1, h_1)$. Therefore $\simeq$ is a symmetric relation, so $\simeq$ is an <u>equivalence relation</u> on any set of compactifications of X.

Example 4.2 Suppose X is a locally compact, noncompact Hausdorff space. We claim that all one-point compactifications of X are equivalent. Because $\simeq$ is transitive, it is sufficient to show that each one-point compactification (Y_1, h_1) is equivalent to the one-point compactification (Y^*, i) constructed in Theorem 2.3.

Let $Y^* = X \cup \{p\}$ and $Y_1 - h_1[X] = \{p_1\}$. Define $f : Y^* \to Y_1$ by

$$f(y) = \begin{cases} h_1(y) & \text{if } y \in X \\ p_1 & \text{if } y = p \end{cases}$$

f is clearly a bijection and $f \circ i = h_1$. We claim f is continuous.

> If $y \in X$: Let V be an open set in Y_1 with $f(y) = h_1(y) \in V$. Then $V' = V - \{p_1\}$ is also open in $h_1[X]$. Since $h_1 : X \to h_1[X]$ is a homeomorphism, $U = h_1^{-1}[V_1]$ is open in X and X is open in Y^*. Then $y \in U$, U is open in Y^* and $f[U] = h_1[U] = V' \subseteq V$. Therefore f is continuous at y.
>
> If $y = p$: Let V be an open set in Y_1 with $f(p) = p_1 \in V$. Then $Y_1 - V = K_1$ is a compact in $h_1[X]$, so $h_1^{-1}[K_1] = K$ is a compact (therefore closed) set in Y^*. Then $U = Y^* - K$ is a neighborhood of p and $f[U] \subseteq V$. Therefore f is continuous at p.

Since f is a continuous bijection from a compact space to a T_2 space, f is closed and therefore f is a homeomorphism.

Therefore (up to equivalence) we can talk about the one-point compactification of a noncompact, locally compact Hausdorff space X. Topologically, it makes no difference whether we think of the one-point compactification of $\mathbb{R}$ geometrically as S^1, with the North Pole p as the "point at infinity," or whether we think of it more abstractly as the result of the construction in Theorem 2.3.

Question: Are all two point compactifications of $(-1, 1)$ equivalent to $[-1, 1]$?

Example 4.3 Suppose (Y_1, h_1) is a compactification of X. Then (Y_1, h_1) is equivalent to a compactification (Y, i) where $X \subseteq Y$ and i is the identity map. We simply define $Y = (Y_1 - h_1[X]) \cup X$, topologized in the obvious way – in effect, we are simply giving each point $h_1(x)$ in Y_1 a new "name" x. We can then define $f : Y_1 \to Y$ by

$$f(z) = \begin{cases} z & \text{if } z \in Y_1 - h_1[X] \\ i \circ h_1^{-1}(z) = h_1^{-1}(z) & \text{if } z \in h_1[X] \end{cases}$$

Clearly, $f \circ h_1 = i$, so $(Y_1, h_1) \simeq (Y, i)$.

Example 4.3 shows means that whenever we work with only one compactification of X, or are discussing properties that are shared by all equivalent compactifications of X, we might as well (for simplicity) replace (Y_1, h_1) with an equivalent compactification Y where Y contains X as a dense subspace.

Example 4.4 Homeomorphic compactifications are not necessarily equivalent. In this example we see two dense embeddings h_1, h_2 of $\mathbb{N}$ into the same compact Hausdorff space Y that produce nonequivalent compactifications.

Let $Y = \{(\frac{1}{n}, i) : i = 1, 2 \text{ and } n \in \mathbb{N}\} \cup \{(0, 1),\ (0, 2)\} \subseteq \mathbb{R}^2$.

Let $h_1 : \mathbb{N} \to Y$ by $\begin{cases} h_1(2n) & = (\frac{1}{n}, 1) \\ h_1(2n-1) & = (\frac{1}{n}, 2) \end{cases}$. (Y, h_1) is a 2-point compactification of $\mathbb{N}$.

Let $h_2 : \mathbb{N} \to Y$ by $\begin{cases} h_2(n) = (\frac{1}{j}, 1) & \text{if } n \text{ is the } j^{\text{th}} \text{ element of } \{1, 2, 4, 5, 7, 8, 10, 11, ...\} \\ h_2(n) = (\frac{1}{j}, 2) & \text{if } n \text{ is the } j^{\text{th}} \text{ element of } \{3, 6, 9, 12, ...\} \end{cases}$

For example, $h_2(7) = (\frac{1}{5}, 1)$ and $h_2(9) = (\frac{1}{3}, 2)$. (Y, h_2) is also a two-point compactification of $\mathbb{N}$.

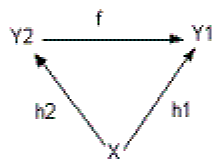
Topologically, each compactification is the same space Y, but (Y, h_1) and (Y, h_2) are <u>not equivalent</u> compactifications of $\mathbb{N}$:

> Suppose $f : Y \to Y$ is any (onto) homeomorphism.
>
> $(h_1(2n)) \to (0, 1)$, so $f(h_1(2n)) \to f((0, 1))$, and $f((0, 1)) =$ <u>either</u> $(0, 1)$ <u>or</u> $(0, 2)$ (*why?*).
>
> But the sequence $(h_2(2n)) = ((\frac{1}{2}, 1), (\frac{1}{3}, 1), (\frac{1}{2}, 2), (\frac{1}{6}, 1), (\frac{1}{7}, 1), (\frac{1}{4}, 2), \dots)$. does <u>not</u> converge to either $(0, 1)$ or $(0, 2)$. Therefore $f \circ h_1 \neq h_2$, so $(Y_1, h_1) \not\simeq (Y_2, h_2)$

By adjusting the definitions of h_1 and h_2, we can create infinitely many nonequivalent 2-point compactifications of $\mathbb{N}$ all using different embeddings of $\mathbb{N}$ into the same space Y.

We now define a relation $\geq$ between compactifications of a space X.

Definition 4.5 Suppose (Y_2, h_2) and (Y_1, h_1) are compactifications of X. We say that $(Y_2, h_2) \geq (Y_1, h_1)$ if there exists a continuous function $f : Y_2 \to Y_1$ such that $f \circ h_2 = h_1$.



Notice that:

i) Such a mapping f <u>is necessarily onto</u> Y_1: $f[Y_2]$ is compact and therefore closed in Y_1; so $f[Y_2] = \text{cl}\, f[Y_2] \supseteq \text{cl}\, f[h_2[X]] = \text{cl}\, h_1[X] = Y_1$.

ii) If $X \subseteq Y_2$, $X \subseteq Y_1$ and $h_1 = h_2 =$ the identity map i, then the condition $f \circ h_2 = h_1$ simply states that $f|X = i$.

iii) $f[Y_2 - h_2[X]] \subseteq Y_1 - h_1[X]$: that is, the "points added" to create Y_2 are mapped onto the "points added" to create Y_1. To see this, let $z \in Y_2 - h_2[X]$. We want to show $f(z) \in Y_1 - h_1[X]$. So suppose that $f(z) = h_1(x) \in h_1[X]$.

Since $h_2[X]$ is dense in Y_2, there is a net in $h_2[X]$ converging to z :

$$(h_2(x_\lambda)) \to z \qquad (*)$$

f is continuous, so $\quad h_1(x_\lambda) = f(h_2(x_\lambda)) \to f(z) = h_1(x)$.
But $h_1 : X \to h_1[X]$ is a homeomorphism so

$$(x_\lambda) = (h_1^{-1} h_1(x_\lambda)) \to h_1^{-1} h_1(x) = x \in X,$$

and therefore $\quad (h_2(x_\lambda)) \to h_2(x) \in h_2[X] \qquad (**)$

A net in Y_2 has at most one limit, so $(*)$ and $(**)$ give that $z = h_2(x)$. This is impossible since $z \notin h_2[X]$.

iv) From iii) we conclude that if $(Y_2, h_2) \geq (Y_1, h_1)$, then $|Y_2 - h_2[X]| \geq |Y_1 - h_1[X]|$

Suppose $(Y_2, h_2) \geq (Y_1, h_1)$. The next theorem tells us that the relation " $\geq$ " is unaffected if we replace these compactifications of X with equivalent compactifications – so we can actually compare equivalence classes of compactifications of X by comparing representatives of the equivalence classes. The proof is very easy and is omitted.

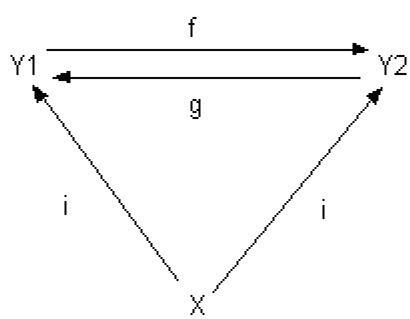
Theorem 4.6 Suppose (Y_2, h_2) and (Y_1, h_1) are compactifications of X and that $(Y_2, h_2) \geq (Y_1, h_1)$. If $(Y_2, h_2) \simeq (Y_2', h_2')$ and $(Y_1, h_1) \simeq (Y_1', h_1')$, then $(Y_2', h_2') \geq (Y_1', h_1')$.

The ordering " $\geq$ " is well behaved on the equivalence classes of compactifications of X.

Theorem 4.7 Let $\mathcal{C}$ be a set of equivalence classes of compactifications of X. Then $(\mathcal{C}, \geq)$ is a poset.

Proof It is clear from the definition that $\geq$ is both reflexive and transitive. We need to show that $\geq$ is also antisymmetric. Suppose $[(Y_1, i)]$ and $[(Y_2, i)]$ are equivalence classes of compactifications of X (*By Theorem 4.6, we are free to choose from each equivalence class representative compactifications with* $X \subseteq Y_i$ *and embeddings* $h = i =$ *the identity map*).

If both $(Y_1, i) \geq (Y_2, i)$ and $(Y_2, i) \geq (Y_1, i)$ hold, then we have the following maps:



with $f \circ i = i = g \circ i$. For $x \in X$, $g(f(x)) = g(f(i(x)) = g(i(x)) = i(x) = x$, so the maps $g \circ f$ and the identity $i : Y_1 \to Y_1$ agree on the dense subspace X. Since Y_1 is Hausdorff, it follows that $g \circ f = i$ everywhere in Y_1. (*See Theorem 5.12 in Chapter II, and its generalization in Exercise E9 of Chapter III.*) Similarly $f \circ g$ and $i : Y_2 \to Y_2$ agree on the dense subspace X so $f \circ g = i$ on Y_2.

Since $f \circ g = i$ and $g \circ f = i$, f and g are inverse functions and f is a homeomorphism. Therefore $(Y_1, i) \simeq (Y_2, i)$. So we have shown that if $[(Y_1, i)] \geq [(Y_2, i)]$ and $[(Y_1, i)] \leq [(Y_2, i]$, then $[(Y_1, i)] = [(Y_2, i]$ $\bullet$

An equivalence class of compactifications of a space X is "too big" to be a set in ZFC set theory. (It is customary to refer informally to such collections "too big" to be sets in ZFC as "classes.")

However, suppose (Y, i) represents one of these equivalence classes. If X has weight m, then X contains a dense set D with $|D| \leq m$. It follows from Lemma 3.3 that $w(Y) \leq 2^m$ so, by Theorem VII.3.17, Y can be embedded in the cube $[0,1]^{2^m}$. Therefore every compactification of X can be represented by a subspace of the one fixed cube $[0,1]^{2^m}$.

Therefore we can form a set consisting of one representative from each equivalence class of compactifications of X : this set is just a certain set of subspaces of $[0,1]^{2^m}$. This set is partially ordered by $\geq$.

In fact, we can even given a bound on the number of different equivalence classes of compactifications of X: since every compactification of X can be represented as a subspace of $[0,1]^{2^m}$, the number of equivalence classes of compactifications of X is no more than $|\mathcal{P}([0,1]^{2^m})| = 2^{(c^{2^m})} = 2^{2^{2^m}}$. In other words, there are no more than $2^{2^{2^m}}$ different compactifications of X.

Example 4.8 Let (Y_1, h_1) be a 1-point compactification of X. For every compactification (Y, h) of X, $(Y, h) \geq (Y_1, h_1)$. (*So, among equivalence classes of compactifications of X, the equivalence class $[(Y_1, h_1)]$ is smallest.*)

By Theorem 4.6, we may assume $X \subseteq Y_1$, $X \subseteq Y$ and that h, h_1 are the identity maps; in fact, we may as well assume $Y_1 = X^*$ (the one-point compactification constructed in Theorem 2.3).

Since X has a one-point compactification, X is locally compact (*see Example 1.5*). By Corollary 2.7, X is open in both Y and X^*.

Let $X^* - X = \{p\}$ and define

$$f : Y \to X^* \text{ by } f(y) = \begin{cases} y & \text{if } y \in X \\ p & \text{if } z \in Y - X \end{cases}$$

To show that $(Y, i) \geq (X^*, i)$, we only need to check that f is continuous each point $z \in Y$.

> If $y \in X$ and V is an open set containing $f(y) = y$ in X^*, then $y \in U = V - \{p\}$ which is open in X and therefore also open in Y. Clearly, $f[U] = U \subseteq V$.
>
> If $z \in Y - X$ and V is an open neighborhood of $f(z) = p$ in X^*, then $X^* - V = K$ is a compact subset of X. Therefore K is closed in Y so $U = Y - K$ is an open set in Y containing z and $f[U] \subseteq V$.

5. The Stone-Cech Compactification

Example 4.8 shows that the one-point compactification of a space X, when it exists, is the smallest compactification of X. Perhaps it is surprising that every Tychonoff space X has a largest compactification and, by Theorem 4.7, this compactification is unique up to equivalence. In other words, a poset which consists of one representative of each equivalence class of compactifications of X has a largest (not merely maximal!) element. This largest compactification of X is called the Stone-Cech (pronounced "check") compactification and is denoted by βX.

Theorem 5.1 1) Every Tychonoff Space X has a largest compactification, and this compactification is unique up to equivalence. (*We may represent the largest compactification by* $(\beta X, i)$ *where* $\beta X \supseteq X$ *and* i *is the identity map. We do this in the remaining parts of theorem.*)

2) Suppose X is Tychonoff and that Y is a compact Hausdorff space. Every continuous $f : X \to Y$ has a unique continuous extension $f^\beta : \beta X \to Y$. (*The extension* f^β *is called the Stone extension of* f. *The property of* βX *in 2) is called the Stone Extension Property.*)

3) Up to equivalence, βX is the only compactification of X with the Stone Extension Property. (*In other words, the Stone Extension Property characterizes* βX *among all compactifications of* X.)

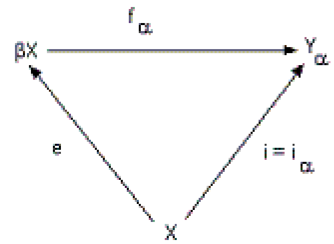
Example 5.2 (assuming Theorem 5.1) $[0,1]$ is a compactification of $(0,1]$. However the continuous function $f : (0,1] \to Y = [-1,1]$ given by $f(x) = \sin(\frac{1}{x})$ cannot be continuously extended to a map $f^\beta : [0,1] \to Y$. Therefore $[0,1] \neq \beta(0,1]$. Is it possible that $S^1 = \beta\mathbb{R}$?

Proof of Theorem 5.1

1) Since X is Tychonoff, X has at least one compactification. Let $\{(Y_\alpha, i_\alpha) : \alpha \in A\}$ be a set of compactifications of X, where $X \subseteq Y_\alpha$, $i_\alpha : X \to Y_\alpha$ is the identity, and Y_α is chosen from each equivalence class of compactifications of X. (*As noted in the remarks following Theorem 4.7, this is a legitimate set since every compactification of* X *can be represented as subset of one fixed cube* $[0,1]^k$.)

Define $e : X \to Y = \prod\{Y_\alpha : \alpha \in A\}$ by $e(x)(\alpha) = i(x) = x$. This "diagonal" map e sends each x to the point in the product all of whose coordinates are x, and e is the evaluation map generated by the collection of maps $i_\alpha : X \to Y_\alpha$. X is a subspace of Y_α, and the subspace topology is precisely the weak topology induced on X by each i_α (*see Example VI.2.5*). It follows from Theorem VI.4.4 that e is an embedding of X into the compact space Y. If we define $\beta X = \mathrm{cl}_Y\, e[X]$, then $(\beta X, e)$ is a compactification of X.

Every compactification of X is equivalent to one of the (Y_α, i_α). Therefore, to show βX is the largest compactification we need only show that $(\beta X, e) \geq (Y_\alpha, i_\alpha)$ for each $\alpha \in A$. This, however, is clear: in the diagram below, simply let $f_\alpha = \pi_\alpha | \beta X$.



Then $f_\alpha \circ e = i$ because $f_\alpha(e(x)) = \pi_\alpha(e(x)) = e(x)(\alpha) = i(x) = x$.

Therefore $(\beta X, e) \geq (Y_\alpha, i)$.

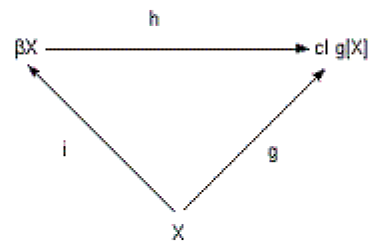
Note: now that the construction is complete, we can replace $(\beta X, e)$ *with an equivalent largest compactification actually containing* X *:* $(\beta X, i)$.

Since $\geq$ is antisymmetric among the compactifications (Y_α, i_α), the largest compactification of X is unique (up to equivalence).

2) Suppose $f : X \to Y$ where Y is a compact Hausdorff space. First, we need to produce a continuous extension $f^\beta : \beta X \to Y$.

Define $g : X \to \beta X \times Y$ by $g(x) = (x, f(x))$. Clearly, g is $1-1$ and continuous, and X has the weak topology generated by the maps $i : X \to X$ and $f : X \to Y$, so g is an embedding. Since $\beta X \times Y$ is compact, $(\mathrm{cl}_{\beta X \times Y}\, g[X],\ g)$ is a compactification of X.

But $(\beta X, i) \geq (\mathrm{cl}\, g[X], g)$, so we have a continuous map $h : \beta X \to \mathrm{cl}\, g[X]$ for which $h \circ i = g$ $-$ that is $h(x) = g(x)$ for $x \in X$ (*see the following diagram*)



For $z \in \beta X$, define $f^\beta(z) = \pi_Y \circ h(z)$. Then f^β is continuous and for $x \in X$ we have $f^\beta(x) = \pi_Y(h(x)) = \pi_Y(h(i(x))) = \pi_Y(g(x)) = \pi_Y(x, f(x)) = f(x)$, so $f^\beta|X = f$.

If $k : \beta X \to Y$ is continuous and $k|X = f$, then k and f^β agree on the dense set X, so $k = f^\beta$ Therefore the Stone extension f^β is unique. (*See Theorem II.5.12, and its generalization in exercise E9 of Chapter III.*.)

3) Suppose (Y, i) is a compactification of X with the Stone Extension Property. Then the identity map $i : X \to \beta X$ has an extension $i^* : Y \to \beta X$ such that $i^* \circ i = i$, so $(Y, i) \geq (\beta X, i)$. Since βX is the largest compactification of X, $(Y, i) \simeq (\beta X, i)$. ●

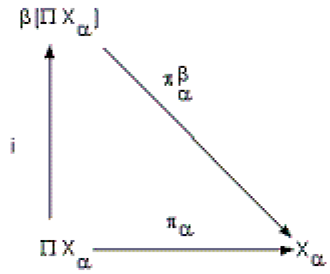
The Tychonoff Product Theorem is equivalent to the Axiom of Choice AC (*see Theorem IX.6.5*). Our construction of βX used the Tychonoff Product Theorem – but only applied to a collection of compact Hausdorff spaces. In fact, as we show below, the existence of a largest compactification βX is equivalent to "the Tychonoff Product Theorem for compact T_2 spaces."

The "Tychonoff Product Theorem for compact T_2 spaces" also cannot be proven in ZF, but it is strictly weaker than AC. (*In fact, the "Tychonoff Product Theorem for compact T_2 spaces" is equivalent to a statement called the "Boolean Prime Ideal Theorem."*)

The main point is that the very existence of βX involves set-theoretic issues and any method for constructing βX must, in some form, use something beyond ZF set theory – something quite close to the Axiom of Choice.

Theorem 5.3 If every Tychonoff space X has of a largest compactification βX, then any product of compact Hausdorff spaces is compact.

Proof Suppose $\{X_\alpha : \alpha \in A\}$ is a collection of compact T_2 spaces. Since $\prod_{\alpha\in A} X_\alpha$ is Tychonoff, it has a compactification $\beta\,[\prod_{\alpha\in A} X_\alpha]$ and for each α the projection map π_α can be extended to $\pi_\alpha^\beta : \beta\,[\prod_{\alpha\in A} X_\alpha] \to X_\alpha$.



For each $x \in \beta\,[\prod_{\alpha\in A} X_\alpha]$, define a point $f(x) \in X$ with coordinates $f(x)(\alpha) = \pi_\alpha^\beta(x)$.

$$f : \beta[\prod_{\alpha\in A} X_\alpha] \to \prod_{\alpha\in A} X_\alpha$$

f is continuous because each coordinate function π_α^β is continuous. If $x \in \prod_{\alpha\in A} X_\alpha$ $\subseteq \beta[\prod_{\alpha\in A} X_\alpha]$, then $f(x)(\alpha) = \pi_\alpha^\beta(x) = \pi_\alpha(x) = x(\alpha) = x_\alpha$ for each α, so $f(x) = x$. Therefore $\prod_{\alpha\in A} X_\alpha$ is a continuous image of the compact space $\beta\,[\prod_{\alpha\in A} X_\alpha]$, so $\prod_{\alpha\in A} X_\alpha$ is compact. •

We want to consider some other ways to recognize βX. Since βX can be characterized by the Stone Extension Property, the following technical theorem about extending continuous functions will be useful.

Theorem 5.4 (Taimonov) Suppose C is a dense subspace of a Tychonoff space X and let Y be a compact Hausdorff space. A continuous function $f : C \to Y$ has a continuous extension $\widetilde{f} : X \to Y$ iff

whenever A and B are disjoint closed sets in Y, $\mathrm{cl}_X f^{-1}[A] \cap \mathrm{cl}_X f^{-1}[B] = \emptyset$.

Proof $\Rightarrow$: If $\widetilde{f}$ exists and A and B are disjoint closed sets in Y, then $\widetilde{f}^{-1}[A] \cap \widetilde{f}^{-1}[B] = \emptyset$. But these sets are closed in X, so $\widetilde{f}^{-1}[A] = \mathrm{cl}_X f^{-1}[A]$ and $\widetilde{f}^{-1}]B] = \mathrm{cl}_X f^{-1}[B]$. Therefore $\mathrm{cl}_X f^{-1}[A] \cap \mathrm{cl}_X f^{-1}[B] = \emptyset$.

$\Leftarrow$: We must define a function $\widetilde{f} : X \to Y$ such that $\widetilde{f}\,|C = f$ and then show that $\widetilde{f}$ is continuous. For $x \in X$, let $\mathcal{N}_x$ be its neighborhood filter in X. Define a collection of closed sets $\mathcal{F}_x$ in Y by

$$\mathcal{F}_x = \{\text{cl } f[C \cap U] : U \in \mathcal{N}_x\}$$

Then $\text{cl } f[C \cap U_1] \cap \text{cl } f[C \cap U_2] \supseteq \text{cl } f[C \cap U_1 \cap U_2] \neq \emptyset$ (since C is dense in X). Therefore $\mathcal{F}_x$ is a family of closed sets in Y with the finite intersection property so $\bigcap \mathcal{F}_x \neq \emptyset$ (because Y is compact) .

We claim that $\bigcap \mathcal{F}_x$ contains only one point: $\bigcap \mathcal{F}_x = \{y\}$ for some $y \in Y$.

> Suppose $y, z \in \bigcap \mathcal{F}_x$. If $y \neq z$, then (since Y is T_3) we can pick open sets U, V so that $y \in U$ and $z \in V$ and $\text{cl } U \cap \text{cl } V = \emptyset$. Then $\text{cl}_X f^{-1}[\text{cl } U] \cap \text{cl}_X f^{-1}[\text{cl } V] = \emptyset$ so, of course, $\text{cl}_X f^{-1}[U] \cap \text{cl}_X f^{-1}[V] = \emptyset$. Taking complements, we get
>
> $$(X - \text{cl}_X f^{-1}[U]) \cup (X - \text{cl}_X f^{-1}[V]) = X$$
>
> so x is in one of these open sets: say $x \in W = X - \text{cl}_X f^{-1}[U]$. Since $W \in \mathcal{N}_x$, $\text{cl } f[C \cap W] \in \mathcal{F}_x$. We claim $\text{cl } f[C \cap W] \subseteq Y - U$, from which will follow the contradiction that $y \notin \bigcap \mathcal{F}_x$.
>
>> To check this inclusion, simply note that $C \cap W = C - \text{cl}_X f^{-1}[U]$. Therefore, if $u \in C \cap W$, we have $u \notin \text{cl}_X f^{-1}[U]$, so $u \notin f^{-1}[U]$, so $f(u) \notin U$. Thus, $f[C \cap W] \subseteq Y - U$ (a closed set) so $\text{cl } f[C \cap W] \subseteq Y - U$.

Define $\widetilde{f}(x) = y$. We claim that $\widetilde{f}$ works.

> $\widetilde{f} \mid C = f$: Suppose $x \in C$. $\mathcal{B} = \{C \cap U : U \in \mathcal{N}_x\}$ is the neighborhood filter of x <u>in C</u> so $\mathcal{B} \to x$ in C. Since f is continuous, the filter base $f[\mathcal{B}] = \{f[C \cap U] : U \in \mathcal{N}_x\}$ $\to f(x)$ in Y. In particular, $f(x)$ is a cluster point of $f[\mathcal{B}]$, so $f(x) \in \bigcap \text{cl}(f[C \cap U])$ $= \bigcap \mathcal{F}_x = \{\widetilde{f}(x)\}$. So $f(x) = \widetilde{f}(x)$.
>
> $\widetilde{f}$ is continuous: Let $x \in X$ and let V be open in Y with $y = \widetilde{f}(x) \in V$. Since $\bigcap \mathcal{F}_x = \{y\} \subseteq V$, there exist $U_1, ..., U_n \in N_x$ such that
>
> $$\text{cl } f[C \cap U_1] \cap ... \cap \text{cl } f[C \cap U_n] \subseteq V$$
>
> (*If V is an open set in a compact space and $\mathcal{F}$ is a family of closed sets with $\bigcap \mathcal{F} \subseteq V$, then some finite subfamily of $\mathcal{F}$ satisfies $F_1 \cap ... \cap F_n \subseteq V$. Why?*)
>
> Let $W = U_1 \cap ... \cap U_n \in \mathcal{N}_x$. If $z \in W$, then
>
> $$\widetilde{f}(z) \in \text{cl } f[C \cap W] \subseteq \text{cl } f[C \cap U_1] \cap ... \cap \text{cl } f[C \cap U_n] \subseteq V$$
>
> so $\widetilde{f}[W] \subseteq V$. Therefore $\widetilde{f}$ is continuous at x. •

Corollary 5.5 Suppose Y_1 and Y_2 are compactification of X where the embeddings are the identity map. Then $(Y_1, i) \simeq (Y_2, i)$ iff : for every pair of disjoint closed sets in X,

$$\text{cl}_{Y_1}A \cap \text{cl}_{Y_1}B = \emptyset \ \Leftrightarrow \text{cl}_{Y_2}A \cap \text{cl}_{Y_2}B = \emptyset) \qquad (*)$$

Proof If $(Y_1, i) \simeq (Y_2, i)$, it is clear that (*) holds. If (*) holds, then Taimonov's Theorem guarantees that the identity maps $i_1 : X \to Y_1$ and $i_2 : X \to Y_2$ can be extended to maps $f_1 : Y_2 \to Y_1$ and $f_2 : Y_1 \to Y_2$. It is clear that $f_1 \circ f_2|X$ and $f_2 \circ f_1|X$ are the identity maps on the dense subspace X. Therefore $f_1 \circ f_2$ and $f_2 \circ f_1$ are each the identity everywhere so f_1, f_2 are homeomorphisms so $(Y_1, i) \simeq (Y_2, i)$. •

For convenience, we repeat here a definition included in the statement of Theorem VII.5.2

Definition 5.6 Suppose A and B are subspaces of X. A and B are completely separated if there exists $f \in C(X)$ such that $f|A = 0$ and $f|B = 1$. (It is easy to see that $0, 1$ can be replaced in the definition by any two real numbers a, b.)

Urysohn's Lemma states that disjoint closed sets in a normal space are completely separated.

Using Taimonov's theorem, we can characterize βX in several different ways. In particular, condition 4) in the following theorem states that βX is actually characterized by the "extendability" of continuous functions from X into $[0,1]$ – a statement which looks weaker than the full Stone Extension Property.

Theorem 5.7 Suppose (Y, i) is a compactification of X, where $Y \supseteq X$ and i is the identity. The following are equivalent:

1) Y is βX (that is, Y is the largest compactification of X)
2) every continuous $f : X \to K$, where K is a compact Hausdorff space, can be extended to a continuous map $\widetilde{f} : Y \to K$
3) every continuous $f : X \to [a, b]$ can be extended to a continuous $\widetilde{f} : Y \to [a, b]$
4) every continuous $f : X \to [0, 1]$ can be extended to a continuous $\widetilde{f} : Y \to [0, 1]$
5) completely separated sets in X have disjoint closures in Y
6) disjoint zero sets in X have disjoint closures in Y
7) if Z_1 and Z_2 are zero sets in X, then $\text{cl}_Y(Z_1 \cap Z_2) = \text{cl}_Y Z_1 \cap \text{cl}_Y Z_2$,

Proof Theorem 5.1 gives that 1) and 2) are equivalent, and the implications 2) $\Rightarrow$ 3) $\Rightarrow$ 4) are trivial.

4) $\Rightarrow$ 5) If A and B are completely separated in X, then there is a continuous $f : X \to [0, 1]$ with $f|A = 0$ and $f|B = 1$. By 4), f extends to a continuous map $\widetilde{f} : Y \to [0, 1]$. Then $\text{cl}_Y A \subseteq \text{cl}_Y \widetilde{f}^{-1}(0) = \widetilde{f}^{-1}(0)$ and $\text{cl}_Y B \subseteq \text{cl}_Y \widetilde{f}^{-1}(1) = \widetilde{f}^{-1}(1)$, so $\text{cl}_Y A \cap \text{cl}_Y B = \emptyset$.

5) $\Rightarrow$ 6) Disjoint zero sets $Z(f)$ and $Z(g)$ in X are completely separated (for example, by the function $h = \frac{f^2}{f^2 + g^2}$) and therefore, by 5), have disjoint closures in Y.

6 $\Rightarrow$ 7) A zero set neighborhood of X is a zero set Z with $x \in \text{int}\, Z$. It is easy to show that in a Tychonoff space X, the zero set neighborhoods of x form a neighborhood base at x (*check this!*).

Suppose Z_1 and Z_2 are zero sets in X. Certainly, $\mathrm{cl}_Y(Z_1 \cap Z_2) \subseteq \mathrm{cl}_Y Z_1 \cap \mathrm{cl}_Y Z_2$, so suppose $x \in \mathrm{cl}_Y Z_1 \cap \mathrm{cl}_Y Z_2$. If V is a zero set neighborhood of x, then $x \in \mathrm{cl}_Y(Z_1 \cap V)$ and $x \in \mathrm{cl}_Y(Z_2 \cap V)$ (*why?*). $Z_1 \cap V$ and $Z_2 \cap V$ are zero sets in X and $x \in \mathrm{cl}_Y(Z_1 \cap V) \cap \mathrm{cl}_Y(Z_2 \cap V)$ so, by 6). $(Z_1 \cap V) \cap (Z_2 \cap V) = Z_1 \cap Z_2 \cap V \neq \emptyset$.

Since every zero set neighborhood V of x intersects $Z_1 \cap Z_2$, and the zero set neighborhoods of x are a neighborhood base, we have $x \in \mathrm{cl}_Y(Z_1 \cap Z_2)$.

7) $\Rightarrow$ 2) Suppose that $f : X \to K$ is continuous. K is T_4 so if A and B are disjoint closed sets in K, there is a continuous $g : K \to [0,1]$ such that $A \subseteq \{x : g(x) = 0\} = Z_1$ and $B \subseteq \{x : g(x) = 1\} = Z_2$.

Then $f^{-1}[A] \subseteq f^{-1}[Z_1]$ and $f^{-1}[B] \subseteq f^{-1}[Z_2]$ and $f^{-1}[Z_1]$ and $f^{-1}[Z_2]$ are disjoint zero sets in X. By 7), $\mathrm{cl}_Y f^{-1}[A] \cap \mathrm{cl}_Y f^{-1}[B] \subseteq \mathrm{cl}_Y f^{-1}[Z_1] \cap \mathrm{cl}_Y f^{-1}[Z_2] = \mathrm{cl}_Y f^{-1}[Z_1 \cap Z_2]$ $= \emptyset$. By Taimonov's Theorem 5.4, f has a continuous extension $\widetilde{f} : Y \to K$. ●

Example 5.8

1) By Theorem VIII.8.8, every continuous function $f : [0, \omega_1) \to [0,1]$ is "constant on a tail" so f can be continuously extended to $\widetilde{f} : [0, \omega_1] \to [0,1]$. By Theorem 5.7, $[0, \omega_1] = \beta[0, \omega_1)$.

In this case the largest compactification of $[0, \omega_1)$ is the same as the smallest compactification – the one-point compactification. Therefore, up to equivalence, $[0, \omega_1]$ is the <u>only</u> compactification of $[0, \omega_1)$.

A similar example of this phenomenon is $T^* = [0, \omega_1] \times [0, \omega_0] = \beta T$, where $T = T^* - \{(\omega_1, \omega_0)\}$ (see the "Tychonoff plank" in Example VIII.8.10 and Exercise VIII.8.11).

2) The one-point compactification $\mathbb{N}^*$ of $\mathbb{N}$ is <u>not</u> $\beta\mathbb{N}$ because the function $f : \mathbb{N} \to \{0,1\}$ given by $f(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$ cannot be continuously extended to $\widetilde{f} : \mathbb{N}^* \to \{0,1\}$. (*Why? It might help think of* $\mathbb{N}$ *(topologically) as* $\{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$.)

Theorem 5.9 βX is metrizable iff X is a compact metrizable space (i.e., iff X is metrizable and $X = \beta X$).

Proof $\Leftarrow$: Trivial

$\Rightarrow$: βX is metrizable $\Rightarrow \begin{cases} X \text{ is metrizable} \Rightarrow X \text{ is } T_4 \\ \beta X \text{ is first countable} \end{cases}$.

If X is not compact, there is a sequence (x_n) in X with $(x_n) \to p \in \beta X - X$. Without loss of generality, we may assume the x_n's are distinct (*why?*).

Let $O = \{x_1, x_3, ..., x_{2n+1}, ...\}$ and $E = \{x_2, x_4, ..., x_{2n}, ...\}$. O and E are disjoint closed sets in X so Urysohn's Lemma gives us a continuous $f : X \to [0,1]$ for which $f|O = 0$ and $f|E = 1$. Let $f^\beta : \beta X \to [0,1]$ be the Stone Extension of f. Then $f^\beta(p) = \lim_{n\to\infty} f^\beta(x_{2n+1})$
$= \lim_{n\to\infty} f(x_{2n+1}) = 0 \neq 1 = \lim_{n\to\infty} f(x_{2n}) = \lim_{n\to\infty} f^\beta(x_{2n}) = f^\beta(p),$ which is impossible. •

6. The space $\beta\mathbb{N}$

The Stone-Cech compactification of $\mathbb{N}$ is a strange and curious space.

Example 6.1 $\beta\mathbb{N}$ is a compact Hausdorff space in which $\mathbb{N}$ is a countable dense set. Since $\beta\mathbb{N}$ is separable, Theorem 3.4 gives us the upper bound $|\beta\mathbb{N}| \leq 2^{2^{\aleph_0}} = 2^c$.

On the other hand, suppose $f : \mathbb{N} \to [0,1] \cap \mathbb{Q}$ is a bijection and consider the Stone extension $f^\beta : \beta\mathbb{N} \to [0,1]$. Since $f^\beta[\beta\mathbb{N}]$ is compact, it is a closed set in $[0,1]$ and it contains the dense set $\mathbb{Q}$. Therefore $f^\beta[\beta\mathbb{N}] = [0,1]$ so we have $c \leq |\beta\mathbb{N}| \leq 2^c$.

A similar argument makes things even clearer. By Pondiczerny's Theorem VI.3.5, there is a countable dense set $D \subseteq [0,1]^{[0,1]}$. Pick a bijection $f : \mathbb{N} \to D$ and consider the extension $f^\beta : \beta\mathbb{N} \to [0,1]^{[0,1]}$. Just as before, f^β must be onto. Therefore $[\beta\mathbb{N}] \geq |[0,1]^{[0,1]}| = c^c = 2^c$.

Combining this with our earlier upper bound, we conclude that $|\beta\mathbb{N}| = 2^c$. $\beta\mathbb{N}$ is quite large but it contains the dense discrete set $\mathbb{N}$ that is merely countable.

Every set $A \subseteq \mathbb{N}$ is a zero set in $\mathbb{N}$ so we can write

$$\beta\mathbb{N} = \mathrm{cl}_{\beta\mathbb{N}}\, \mathbb{N} = \mathrm{cl}_{\beta\mathbb{N}} A \cup \mathrm{cl}_{\beta\mathbb{N}}(\mathbb{N} - A),$$

and by Theorem 5.7(6) these sets are disjoint. Therefore for each $A \subseteq \mathbb{N}$, $\mathrm{cl}_{\beta\mathbb{N}} A$ is a clopen set in $\beta\mathbb{N}$. In particular, each singleton $A = \{n\}$ is open in $\beta\mathbb{N}$ (that is, n is isolated in $\beta\mathbb{N}$), so $\mathbb{N}$ is open in $\beta\mathbb{N}$. Therefore $\beta\mathbb{N} - \mathbb{N}$ is compact.

At each $x \in \beta\mathbb{N}$, there is a neighborhood base $\mathcal{B}_x$ consisting of clopen neighborhoods:

i) if $x \in \mathbb{N}$, we can use $\mathcal{B}_x = \{\{x\}\}$
ii) if $x \in \beta\mathbb{N} - \mathbb{N}$, we can use $\mathcal{B}_x = \{\mathrm{cl}_{\beta\mathbb{N}} A : A \subseteq \mathbb{N} \text{ and } x \in \mathrm{cl}_{\beta\mathbb{N}} A\}$

If U is an open set in $\beta\mathbb{N}$ containing x, we can use regularity to choose an open set W such that $x \in W \subseteq \mathrm{cl}_{\beta\mathbb{N}} W \subseteq U$. If $A = W \cap \mathbb{N}$, then $x \in \mathrm{cl}_{\beta\mathbb{N}} A = \mathrm{cl}_{\beta\mathbb{N}}(W \cap \mathbb{N}) = \mathrm{cl}_{\beta\mathbb{N}} W \subseteq U$. (*Why?*)

Definition 6.2 Suppose $A \subseteq X$. A is said to be C^*-embedded in X if every $f \in C^*(A)$ has a continuous extension $\widetilde{f} \in C^*(X)$.

To illustrate the terminology:

i) Tietze's Theorem states that every closed subspace of a normal space is C^*-embedded.

ii) For a Tychonoff space X, βX is <u>the</u> compactification (up to equivalence) in which X is C^*-embedded.

The following theorem is very useful in working with βX.

Theorem 6.3 Suppose $A \subseteq X \subseteq \beta X$, and that A is C^*-embedded in X. Then $\mathrm{cl}_{\beta X} A = \beta A$.

Proof If $f : A \to [0,1]$ is continuous, then f extends continuously to $\overline{f} : X \to [0,1]$, and, in turn, $\overline{f}$ extends continuously to $\widetilde{f} : \beta X \to [0,1]$. Then $\widetilde{f}\,|\mathrm{cl}_{\beta X} A$ is a continuous extension of f to $\mathrm{cl}_{\beta X} A$. Since $\mathrm{cl}_{\beta X} A$ has the extension property in Theorem 5.7 (4), $\mathrm{cl}_{\beta X} A = \beta A$. •

Example 6.4 Since $\mathbb{N}$ is discrete, every $A \subseteq \mathbb{N}$ is C^*-embedded in $\mathbb{N}$ and so, by Theorem 6.3, $\mathrm{cl}_{\beta\mathbb{N}} A = \beta A$.

Of course if A is finite, $\mathrm{cl}_{\beta\mathbb{N}} A = A = \beta A$. But if A is infinite, then A is homeomorphic to $\mathbb{N}$, so $\mathrm{cl}_{\beta\mathbb{N}} A = \beta A$ is homeomorphic to $\beta\mathbb{N}$.

In particular, if $\mathbb{E}$ and $\mathbb{O}$ are the sets of even and odd natural numbers, we have $\mathbb{N} = \mathbb{E} \cup \mathbb{O}$, so $\beta\mathbb{N} = \mathrm{cl}_{\beta\mathbb{N}}\mathbb{E} \cup \mathrm{cl}_{\beta\mathbb{N}}\mathbb{O}$ – so $\beta\mathbb{N}$ is the union of two disjoint, clopen copies of itself. It is easy to modify this argument to show that, for any natural number k, $\beta\mathbb{N}$ <u>can be written as the union of k disjoint clopen copies of itself</u>.

If we write $\mathbb{N} = \bigcup_{k=1}^{\infty} A_k$, where each A_k's are pairwise disjoint infinite subsets of $\mathbb{N}$, then we have $\beta\mathbb{N} = \mathrm{cl}_{\beta\mathbb{N}} \bigcup_{k=1}^{\infty} A_k \supseteq \bigcup_{k=1}^{\infty} \mathrm{cl}_{\beta\mathbb{N}} A_k$, and these sets $\mathrm{cl}_{\beta\mathbb{N}} A_k$ are pairwise disjoint copies of $\beta\mathbb{N}$. Moreover, $\bigcup_{k=1}^{\infty} \mathrm{cl}_{\beta\mathbb{N}} A_k$ is dense in $\beta\mathbb{N}$ since the union contains $\mathbb{N}$. (*If we choose the A_k's properly chosen, can we have $\beta\mathbb{N} = \bigcup_{k=1}^{\infty} \mathrm{cl}_{\beta\mathbb{N}} A_k$? Why or why not?*)

Example 6.5 No sequence (n_k) in $\mathbb{N}$ can converge to a point of $\beta\mathbb{N} - \mathbb{N}$. In particular, the sequence (n) has no convergent subsequence in $\beta\mathbb{N}$ so $\beta\mathbb{N}$ is not sequentially compact.

Define $f : \mathbb{N} \to \{0,1\}$ by $f(x) = \begin{cases} 1 & \text{if } x = n_{2k} \\ 0 & \text{otherwise} \end{cases}$. Consider the Stone extension $f^{\beta} : \beta\mathbb{N} \to \{0,1\}$. If $(n_k) \to p \in \beta\mathbb{N} - \mathbb{N}$, then $(f^{\beta}(n_k)) = (f(n_k)) \to f^{\beta}(p) \in \{0,1\}$, so $(f(n_k))$ must be eventually constant – which is false.

Therefore $\beta\mathbb{N}$ is an example showing that "compact $\not\Rightarrow$ sequentially compact." (*See the remarks before and after corollary VIII.8.5.*)

Theorem 6.6 Every infinite closed set F in $\beta\mathbb{N}$ contains a copy of $\beta\mathbb{N}$ and therefore satisfies $|F| = 2^c$.

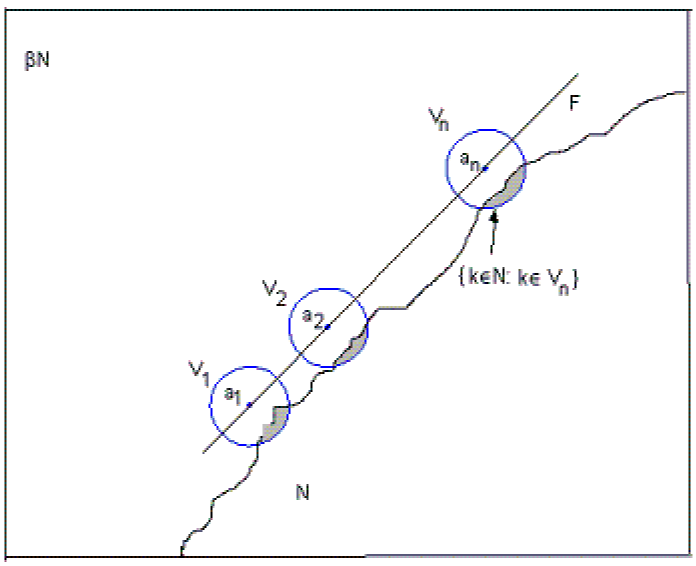
Proof Pick an infinite discrete set $A = \{a_n : n = 1, 2, ...\} \subseteq F$. (*See Exercise III E9*). Using regularity, pick pairwise disjoint open sets V_n in $\beta\mathbb{N}$ with $a_n \in V_n$.

Suppose $g : A \to [0, 1]$ (g is continuous since A is discrete). Define $G : \mathbb{N} \to [0, 1]$ by

$$G(k) = \begin{cases} g(a_n) & \text{for } k \in \mathbb{N} \cap V_n \\ 0 & \text{for } k \in \mathbb{N} - \bigcup_{n=1}^{\infty} V_n \end{cases}$$

Extend G to a continuous map $G^\beta : \beta\mathbb{N} \to [0, 1]$.

The following diagram gives a very "distorted" image of how the sets in the argument are related.



We have $G^\beta | \mathbb{N} \cap V_n = g(a_n)$. Since $\mathbb{N} \cap V_n$ is dense in V_n (*why?*), we have $G^\beta | V_n = g(a_n)$ so $G^\beta | A = g$.

Thus, $g : A \to [0, 1]$ has an extension $G^\beta : \beta\mathbb{N} \to [0, 1]$, so A is C^*-embedded in $\beta\mathbb{N}$. By Theorem 6.3, $\text{cl}_{\beta\mathbb{N}} A = \beta A$ and since A is a countably infinite discrete space, βA is homeomorphic to $\beta\mathbb{N}$.

Since F is closed, $\text{cl}_{\beta\mathbb{N}} A = \beta A \subseteq F$, so $|F| = 2^c$. •

Theorem 6.6 illustrates a curious property of $\beta\mathbb{N}$: there is a "gap" in the sizes of closed subsets. That is, every closed set in $\beta\mathbb{N}$ is either finite or has cardinality 2^c – no sizes in-between! This "gap in the possible sizes of closed subsets" can sometimes occur, however, even in spaces as nice as metric spaces – although not if the Generalized Continuum Hypothesis is assumed. (See A.H. Stone, *Cardinals of Closed Sets*, Mathematika 6 (1959), pp. 99-107.)

Example 6.7 $\beta\mathbb{N}$ is separable, but its subspace $\beta\mathbb{N}-\mathbb{N}$ is not; $\beta\mathbb{N}-\mathbb{N}$ does not even satisfy the weaker countable chain condition CCC (see Definition VIII.11.4). Specifically, we will show that $\beta\mathbb{N}-\mathbb{N}$ contains c pairwise disjoint clopen (in $\beta\mathbb{N}-\mathbb{N}$) subsets, each of which is homeomorphic to $\beta\mathbb{N}-\mathbb{N}$.

Let $\{N_t : t \in [0,1]\}$ be a collection of c infinite subsets of $\mathbb{N}$ with the property that any two have finite intersection. (*See Exercise I.E41.*) Let $U_t = (\beta\mathbb{N}-\mathbb{N}) \cap \mathrm{cl}_{\beta\mathbb{N}} N_t = \mathrm{cl}_{\beta\mathbb{N}} N_t - N_t$. Each $U_t \neq \emptyset$ (*why?*) and U_t is a clopen set in $\beta\mathbb{N}-\mathbb{N}$ homeomorphic to $\beta\mathbb{N}-\mathbb{N}$.

Moreover, the U_t's are disjoint:

> Suppose $t \neq t'$. If $z \in U_t \cap U_{t'}$, then $z \in \mathrm{cl}_{\beta\mathbb{N}} N_t \cap \mathrm{cl}_{\beta\mathbb{N}} N_{t'}$. In a T_1 space, deleting finitely many points from an infinite set A does not change the set $\mathrm{cl}\, A - A$ (*why?*), so $z \in \mathrm{cl}_{\beta\mathbb{N}}(N_t - (N_t \cap N_{t'}))$ and $z \in \mathrm{cl}_{\beta\mathbb{N}}(N_{t'} - (N_t \cap N_{t'}))$. But $N_t - (N_t \cap N_{t'})$ and $N_{t'} - (N_t \cap N_{t'})$ are disjoint zero sets in $\mathbb{N}$ and must have disjoint closures.

An additional tangential observation:

If we choose points $x_t \in U_t$ and let $X = \mathbb{N} \cup \{x_t : t \in [0,1]\}$, then X is not normal – since a separable normal space cannot have a closed discrete subset $\{x_t : t \in [0,1]\}$ of cardinality c. (See the "counting continuous functions" argument in Example VII.5.6.)

The following example shows us that countable compactness and pseudocompactness are not even finitely productive.

Example 6.8 There is a countably compact space X for which $X \times X$ is not pseudocompact (so $X \times X$ is also not countably compact).

Let $\mathbb{E} = \{2,4,6,...\}$ and $\mathbb{O} = \{1,3,5,...\}$ and write $\beta\mathbb{N} = \mathrm{cl}_{\beta\mathbb{N}}\mathbb{E} \cup \mathrm{cl}_{\beta\mathbb{N}}\mathbb{O} = \beta\mathbb{E} \cup \beta\mathbb{O}$.
$\beta\mathbb{E}$ and $\beta\mathbb{O}$ are disjoint clopen copies of $\beta\mathbb{N}$. Choose any homeomorphism $f : \beta\mathbb{E} \longrightarrow \beta\mathbb{O}$ (n*ecessarily, $f[\mathbb{E}] = \mathbb{O}$: why?*) and define $g : \beta\mathbb{N} \to \beta\mathbb{N}$ by $g(x) = \begin{cases} f(x) & \text{if } x \in \beta\mathbb{E} \\ f^{-1}(x) & \text{if } x \in \beta\mathbb{O} \end{cases}$.

The map g is a homeomorphism since g and g^{-1} are continuous on the two disjoint clopen sets $\beta\mathbb{E}$ and $\beta\mathbb{O}$ whose union is $\beta\mathbb{N}$. Clearly, $g|\mathbb{N} : \mathbb{N} \to \mathbb{N}$, g has no fixed points, and $g \circ g$ is the identity map.

Let $\mathcal{C} = \{A \subseteq \beta\mathbb{N} : A \text{ is countably infinite}\}$. $|\,\mathcal{C}\,| = (2^c)^{\aleph_0} = 2^c$. Let λ be the first ordinal with cardinality 2^c and index $\mathcal{C}$ as $\{A_\alpha : \alpha < \lambda\}$. For each α, $|\mathrm{cl}_{\beta\mathbb{N}} A_\alpha|$ is an infinite closed set so, by Theorem 6.6, $|\mathrm{cl}_{\beta\mathbb{N}} A_\alpha| = 2^c$. Therefore $\mathrm{cl}_{\beta\mathbb{N}} A_\alpha - A_\alpha \neq \emptyset$.

Pick p_0 to be a limit point of A_0 not in A_0. Proceeding inductively, assume that for all $\alpha < \beta < \lambda$ we have chosen a limit point p_α of A_α that is not in A_α and that, for the points p_α, p_γ $(\alpha < \gamma < \beta)$ already defined :

$$\begin{cases} p_\alpha \neq p_\gamma \\ p_\alpha \neq g(p_\gamma) \\ p_\gamma \neq g(p_\alpha) \end{cases} \qquad (*)$$

For the “next step”, we want to define p_β . Since $|[0,\beta)| < 2^c$, we have so far defined fewer than 2^c points p_α. Therefore

$$|\{p_\alpha : \alpha < \beta\} \cup \{g(p_\alpha) : \alpha < \beta\} \cup \{g^{-1}(p_\alpha) : \alpha < \beta\}| < 2^c.$$

But $|\mathrm{cl}_{\beta\mathbb{N}} A_\beta - A_\beta| = 2^c$, so we can chose a limit point p_β of A_β with $p_\beta \notin A_\beta$ so that the conditions $(*)$ continue to hold for $\alpha < \gamma < \beta + 1$.

Therefore, by transfinite recursion, we have defined distinct points p_α $(\alpha < \lambda)$ in such a way that for $\alpha \neq \beta < \lambda,\ g(p_\alpha) \neq p_\beta$ and $g(p_\beta) \neq p_\alpha$.

Let $X = \mathbb{N} \cup \{p_\alpha : \alpha < \lambda\}$. By construction, X is countably compact because every infinite set in X (for that matter, even every infinite set in $\beta\mathbb{N}$) has a limit point in X. But we claim that $X \times X$ is not pseudocompact.

> To see this, consider $Z = \{(n, g(n) : n \in \mathbb{N}\} \subseteq X \times X$. We claim Z is clopen in $X \times X$.
>
> > Since $(n, g(n))$ is isolated in $X \times X$, Z is a discrete open subset of $X \times X$.
> >
> > On the other hand, the graph of $g = \{(x, g(x)) : x \in \beta\mathbb{N}\}$ is closed in $\beta\mathbb{N} \times \beta\mathbb{N}$ so that
> >
> > > $\{(x, g(x)) : x \in \beta\mathbb{N}\} \cap (X \times X)$ is closed in $X \times X$.
> >
> > and we claim that $\{(x, g(x) : x \in \beta\mathbb{N}\} \cap (X \times X) = Z$.
> >
> > Indeed, it is clear that
> >
> > > $Z \subseteq \{(x, g(x) : x \in \beta\mathbb{N}\} \cap (X \times X)$
> >
> > and the complicated construction of the p_α's was done precisely to guarantee the reverse inclusion:
> >
> > > If $(x, g(x)) \in X \times X$, then $x \in \mathbb{N}$ – for otherwise we would have $x = p_\alpha$ for some α, and then $g(x) = g(p_\alpha) \notin X$ by construction.
> >
> > Therefore Z is closed in $X \times X$.
>
> Therefore function $h : X \times X \to \mathbb{N}$ defined by
>
> $$h(u) = \begin{cases} n & \text{if } u = (n, g(n)) \in Z \\ 0 & \text{if } u \in (X \times X) - Z \end{cases}$$
>
> continuous. But h is unbounded, so $X \times X$ is not pseudocompact.

7. Alternate Constructions of βX

We constructed βX by defining an order $\geq$ between certain compactifications of X and showing that there must exist a largest compactification (unique up to equivalence) in this ordering. Theorem 5.7, however, shows that there are many different characterizations of βX and some of these characterizations suggest other ways to construct βX. .

For example, Theorem 5.7 shows that the zero sets in a Tychonoff space X play a special role in βX. Without going into the details, one can construct βX as follows:

Let $\mathcal{Z}$ be the collection of zero sets in X. A filter $\mathcal{F}$ in $\mathcal{Z}$ (also called a z-filter) means a nonempty collection of nonempty zero sets such that

i) if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$, and
ii) if $F \in \mathcal{F}$ and $G \supseteq F$ where G is a zero set, then $G \in \mathcal{F}$.

A z-ultrafilter in X is a maximal z-filter.

Define a set $\beta X = \{\mathcal{U} : \mathcal{U}$ is a z-ultrafilter in $X\}$. For each $p \in X$, the collection $\mathcal{U}_p = \{Z : Z$ is a zero set containing $p\}$ is a (trivial) z-ultrafilter, so $\mathcal{U}_p \in \beta X$. The map $h(p) = \mathcal{U}_p$ is a $1-1$ map of X into the set βX.

It turns out that X compact iff every z-ultrafilter is of the form $\mathcal{U}_p$ for some $p \in X$. Therefore the set $\beta X - X = \emptyset$ iff X is compact. Each z-ultrafilter $\mathcal{U}$ in X that is not of the trivial form $\mathcal{U}_p$ is a point in $\beta X - X$.

The details of putting a topology on βX to create the largest compactification of X are a bit tricky and we will not go into them here.

The situation is simpler in the case $X = \mathbb{N}$. Since every subset of $\mathbb{N}$ is a zero set, a "z-ultrafilter" in $\mathbb{N}$ is just an ordinary ultrafilter in $\mathbb{N}$.

Then, to be a bit more specific,

let $\beta\mathbb{N} = \{\mathcal{U} : \mathcal{U}$ is an ultrafilter in $\mathbb{N}\}$ and for $A \subseteq \mathbb{N}$, define
cl $A = \{\mathcal{U} : A \in \mathcal{U}\}$

Give $\beta\mathbb{N}$ the topology for which $\{$cl $A : A \subseteq \mathbb{N}\}$ is a base for the open sets.

This topology makes $\beta\mathbb{N}$ into a compact T_2 and we can embed $\mathbb{N}$ into $\beta\mathbb{N}$ using the mapping $h(n) = \mathcal{U}_n$ ($=$ the trivial ultrafilter "fixed" at n). This "copy" of $\mathbb{N}$ is dense in $\beta\mathbb{N}$, so $\beta\mathbb{N}$ is a compactification of $\mathbb{N}$. It can be shown that "this $\beta\mathbb{N}$" is the largest compactification of $\mathbb{N}$ (and therefore equivalent to the $\beta\mathbb{N}$ constructed earlier).

The free ultrafilters in $\mathbb{N}$ are the points in $\beta\mathbb{N} - \mathbb{N}$. Since $|\beta\mathbb{N}| = 2^c$ and there are only countably many trivial ultrafilters $\mathcal{U}_n$, we conclude that there are 2^c free ultrafilters in $\mathbb{N}$

It turns out that the z-ultrafilters in a Tychonoff space X are associated in a natural $1-1$ way with the maximal ideals of the ring $C(X)$, so it is also possible to construct βX by putting an appropriate topology on the set

$$\beta X = \{M : M \text{ is a maximal ideal in } C(X)\}$$

It turns out that if $p \in X$, then $M_p = \{f \in C(X) \ : f(p) = 0\}$ is a (trivial) maximal ideal and the mapping $h(p) = M_p$ gives a natural way to embed X in βX. X is not compact iff there are maximal ideals in $C(X)$ that are not of the form M_p (that is, nontrivial maximal ideals) and these are the points of $\beta X - X$.

More information about these constructions can be found in the beautifully written classic *Rings of Continuous Functions* (Gillman & Jerison).

In this section, we give one alternate construction of βX in detail. It is essentially the construction used by Tychonoff, who was the first to construct βX for arbitrary Tychonoff spaces. In his paper *Über die topologische Erweiterung von Räumen* (Math. Annalen 102(1930), 544-561) Tychonoff also established the notation "βX." The construction involves a specially chosen embedding of X into a cube.

Suppose X is a Tychonoff space. For each $f \in C^*(X)$, choose a closed interval $I_f \subseteq \mathbb{R}$ such that $\text{ran}(f) \subseteq I_f$. If $\mathcal{F} \subseteq C^*(X)$ is a family that $\mathcal{F}$ separates points from closed sets, then according to Theorem VI.4.10 the evaluation map $e_{\mathcal{F}} : X \to \prod_{f\in\mathcal{F}} I_f$ given by $e_{\mathcal{F}}(x) = f(x)$ is an embedding. In this way, every such family $\mathcal{F} \subseteq C^*(X)$ generates a compactification $(\text{cl}\, e_{\mathcal{F}}[X], e_{\mathcal{F}})$ of X. In fact, the following theorem states that every compactification of X can be obtained by choosing the correct family $\mathcal{F} \subseteq C^*(X)$.

Theorem 7.1 Every compactification of X can be achieved using the construction in the preceding paragraph. More precisely, if Y is a compactification containing X (with embedding i), then there exists a family $\mathcal{F} \subseteq C^*(X)$ such that $\mathcal{F}$ separates points and closed sets and $(\text{cl}\, e_{\mathcal{F}}[X], e_{\mathcal{F}}) \simeq (Y, i)$.

Proof Let $\mathcal{F} = \{f \in C^*(X) : f$ can be continuously extended to $\widetilde{f} : Y \to I_f\}$. (*Note that* $\widetilde{f}$ *is unique if it exists* $-$ *since any two extensions would agree on the dense set* X.)

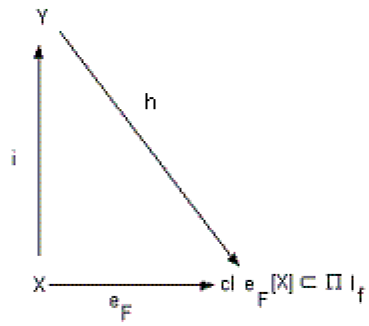
The family $\mathcal{F}$ separates points from closed sets:

If F is a closed set in X and $x \notin F$, then there is a closed set $K \subseteq Y$ with $x \notin K \cap X = F$. By complete regularity there is a continuous function $g : Y \to [0,1]$ such that $g(x) = 0$ and $g|K = 1$. Since Y is compact, g must be bounded and therefore $f = g|X \in C^*(X)$. Moreover, $f \in \mathcal{F}$ (because g is the required extension). Clearly, $f(x) = g(x) = 0 \notin \text{cl}\, f[F] \subseteq \text{cl}\, g[K] = \{1\}$.

Therefore $(\text{cl}\, e_{\mathcal{F}}[X], e_{\mathcal{F}})$ is a compactification of X.

Define $h : Y \to \prod_{f\in\mathcal{F}} I_f$ by $h(p)(f) = \widetilde{f}\,(p)$. Then h is continuous and, for $x \in X$, $h(x)(f)$ $= \widetilde{f}\,(x) = f(x) = e_{\mathcal{F}}(x)(f)$. Therefore $h[X] = e_{\mathcal{F}}[X]$ and $h \circ i = e_{\mathcal{F}}$.

Clearly, $e_{\mathcal{F}}[X] = h[X] \subseteq h[Y]$ and $h[Y]$ is compact Hausdorff, so $\mathrm{cl}\, e_{\mathcal{F}}[X] \subseteq h[Y]$. On the other hand, by continuity, $h[Y] = h[\mathrm{cl}\, X] \subseteq \mathrm{cl}\, h[X] = \mathrm{cl}\, e_{\mathcal{F}}[X]$. Therefore $h[Y] = \mathrm{cl}\, e_{\mathcal{F}}[X]$.



Since $h : Y \to \mathrm{cl}\, e_{\mathcal{F}}[X]$ is continuous and onto, $(Y, i) \geq ((\mathrm{cl}\, e_{\mathcal{F}}[X], e_{\mathcal{F}}))$.

We claim h is also $1-1$:

> If $p \neq q \in Y$, then there is a continuous map $g : Y \to [0,1]$ such that $g(p) = 0$ and $g(q) = 1$. Then $f = g|X \in \mathcal{F}$ and $\widetilde{f}\,(p) = g(p) \neq g(q) = \widetilde{f}\,(q)$. Therefore $h(p)(f) \neq h(q)(f)$, so $h(p) \neq h(q)$.

Since Y is compact and $\mathrm{cl}\, e_{\mathcal{F}}[X]$ is Hausdorff, h is a homeomorphism and, as mentioned above, $h \circ i = e_{\mathcal{F}}$. Therefore $(Y, i) \simeq ((\mathrm{cl}\, e_{\mathcal{F}}[X], e_{\mathcal{F}})$.

Theorem 7.2 Suppose $\mathcal{F} \subseteq \mathcal{F}' \subseteq C^*(X)$ and that both $\mathcal{F}$ and $\mathcal{F}'$ separate points from closed sets. Then $(\mathrm{cl}\, e_{\mathcal{F}'}[X], e_{\mathcal{F}'}) \geq (\mathrm{cl}\, e_{\mathcal{F}}[X], e_{\mathcal{F}})$.

$$\begin{tikzcd}
\mathrm{cl}\, e_{F'}[X] \subset \prod_{f \in F'} I_f \arrow[dr, "h"] & \\
X \arrow[u] \arrow[r] & \mathrm{cl}\, e_{F}[X] \subset \prod_{f \in F} I_f
\end{tikzcd}$$

For $p \in \mathrm{cl}\, e_{\mathcal{F}'}[X]$, define $h(p) \in \mathrm{cl}\, e_{\mathcal{F}}[X]$ by $h(p)(f) = p(f) = p_f$. (*Informally, $h(p)$ is just the result of deleting from p all the coordinates corresponding to functions in $\mathcal{F}' - \mathcal{F}$.*) Clearly $e_{\mathcal{F}} = h \circ e_{\mathcal{F}'}$ so $(\mathrm{cl}\, e_{\mathcal{F}'}[X], e_{\mathcal{F}'}) \geq (\mathrm{cl}\, e_{\mathcal{F}}[X], e_{\mathcal{F}})$. •

Corollary 7.3 A Tychonoff space X has a largest compactification.

Proof Combining Theorems 7.1 and 7.2, we see that the largest compactification corresponds to taking $\mathcal{F} = C^*(X)$ in the preceding construction. •

Of course we can do the construction (from the paragraph preceding Theorem 7.1) simply using $\mathcal{F} = C^*(X)$ in the first place (that is what Tychonoff did) and define the resulting compactification to be βX. We would then need to prove that it has one of the features that make it interesting – for example, the Stone Extension Property. Instead, using Theorems 7.1 and 7.2, what we did was first to argue that $\mathcal{F} = C^*(C)$ produces the largest compactification of X; then Theorem 5.7 told us that the compactification we constructed is the same as our earlier βX.

Exercises

E1. Show that the Sorgenfrey line (Example III.5.3) is not locally compact.

E2. Suppose X is a locally compact T_2 space that is separable and not compact. Show that the one-point compactification X^* is metrizable.

E3. Suppose C and K are disjoint compact subsets in a locally compact Hausdorff space X. Prove that there exist disjoint open sets $U \supseteq C$ and $V \supseteq K$ such that $\operatorname{cl} U$ and $\operatorname{cl} V$ are compact.

E4. a) Let K be a compact subspace of a Tychonoff space X. Prove that for each $g \in C(K)$ there is an $f \in C(X)$ that $g = f|K$ – that is, every continuous real valued function on K can be extended to X. (*A subspace of X with this property is said to be C-embedded in X. Compare Definition 6.2; for a compact since K is compact, "C-embedded" and "C^*-embedded" mean the same thing.*)

b) Suppose A is a dense C-embedded subspace of a Tychonoff space X. If $f \in C(X)$ and $f(x) = 0$ for some $x \in X$, prove that $f(a) = 0$ for some $a \in A$. *Hint: if $f|A$ is never 0, then* $\frac{1}{f} \in C(A)$

c) Every bounded function $f : \mathbb{N} \to \mathbb{R}$ has a continuous extension $f^\beta : \beta\mathbb{N} \to \mathbb{N}$. In particular, the function $f(n) = \frac{1}{n}$ can be extended. If $p \in \beta\mathbb{N} - \mathbb{N}$, what is $f^\beta(p)$? Why does this not contradict part b) ?

E5. Prove that $|\beta\mathbb{R}| = |\beta\mathbb{Q}| = 2^c$.

E6. Prove that a Tychonoff space X is connected iff βX is connected. Is it true that X is connected iff every compactification of X is connected?

E7. a) Show that $\beta\mathbb{R} - \mathbb{R}$ has two components A and B.

b) $[0, \infty)$ has a limit point in $\beta\mathbf{R} - \mathbf{R}$, say in the set B. Is $\beta[0,1) = B$?

E8. Let $\mathcal{U}$ be a free ultrafilter in $\mathbb{N}$.

a) Choose a point $\sigma \in \beta\mathbb{N} - \mathbb{N}$ and let $\mathcal{U} = \{A \subseteq \mathbb{N} : \sigma \in \operatorname{cl}_{\beta\mathbb{N}} A\}$. Show that $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$.

b) Using the ultrafilter $\mathcal{U}$ from a), construct the space Σ as in Exercise IX.E8. Prove that Σ is homeomorphic to $\mathbb{N} \cup \{\sigma\}$ with the subspace topology from $\beta\mathbb{N}$.

c) Define an equivalence relation on $\beta\mathbb{N} - \mathbb{N}$ by $x \sim y$ if $\mathbb{N} \cup \{x\}$ is homeomorphic to $\mathbb{N} \cup \{y\}$. For $x \in \beta\mathbb{N} - \mathbb{N}$, let $[x]$ be the equivalence class of x. Prove that each equivalence class satisfies $|[x]| \leq c$ (so there must be 2^c different equivalence classes.)

Note: Part c) says that, in some sense, there are 2^c topologically different points $\sigma \in \beta\mathbb{N} - \mathbb{N}$. By part a), each of these points σ is associated with a free ultrafilter $\mathcal{U}$ in $\mathbb{N}$ that determines the topology on $\mathbb{N} \cup \{\sigma\}$. Therefore there are 2^c "essentially different" free ultrafilters $\mathcal{U}$ in $\mathbb{N}$.

Chapter X Review

Explain why each statement is true, or provide a counterexample.

1. Every Tychonoff space has a one-point compactification.

2. If X is Tychonoff and βX is first countable, then $|\ \beta X\ | \ \leq c$.

3. $\mathbb{R}$ has a compactification of cardinal 2^{2^c}.

4. $\mathbb{R}$ has a compactification $Y \supseteq \mathbb{R}$ where $Y - \mathbb{R}$ is infinite and Y is metrizable.

5. Suppose that X is a compact Hausdorff space and that each $x \in X$ has a metrizable neighborhood (i.e., X is *locally metrizable*). Then X is metrizable.

6. Let $\mathbb{N}^*$ be the 1-point compactification of $\mathbb{N}$. Every subset of $\mathbb{N}^*$ is Borel.

7. $\beta\mathbb{N} - \mathbb{N}$ is dense in $\beta\mathbb{N}$.

8. If $X = [0, \omega_0 + \omega_0)$, then $\beta X = [0, \omega_0 + \omega_0]$.

9. Every point in $\beta\mathbb{N}$ is the limit of a sequence from $\mathbb{N}$.

10. The one-point compactification of $\mathbb{R}$ is completely metrizable.

11. If X and Y are locally compact Hausdorff spaces with homeomorphic one-point compactifications, then X must be homeomorphic to Y.

12. Let $n \in \mathbb{N}$. All n-point compactifications of the Tychonoff space X are equivalent.

13. Every subset of $\mathbb{R}$ is C^*-embedded in $\mathbb{R}$.

14. If X is compact Hausdorff and $a \in X$, then $\beta(X - \{a\}) = X$.

15. Every compact Hausdorff space is separable.

16. A metric space (X, d) has a metrizable compactification iff X is separable.

17. $\mathbb{Q} = U \cap F$ for some open U and closed F in $\mathbb{R}$.