Right and Left Inverses for a Matrix

D is called a **right inverse** for a $m \times n$ matrix A if $AD = I_m$
($so\ D\ must\ be\ n \times m$). For example, if $A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 4 \end{bmatrix}$, then a right inverse for A is $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ because $AD = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$.

But if $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $AE = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ also, so E is another right inverse for A.

If A has a right inverse, it is not necessarily unique.

C is called a **left inverse** for a $m \times n$ matrix A if $CA = I_n$
($so\ C\ must\ be\ n \times m$)

It turns out that the matrix A above has no left inverse (*see below).* This is no accident ! The following theorem says that if A has both a right and a left inverse, then A must be square.

Theorem If A is $m \times n$ and if

i) D is a a right inverse for A (so $AD = I_m$) and
ii) C is a left inverse for A (so $CA = I_n$)

then $m = n$ (so A is square). Moreover, A is invertible and $A^{-1} = C = D$.

Proof Suppose A is $m \times n$

If $AD = I_m$, then the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for every possible $\boldsymbol{b}$ in $\mathbb{R}^m$ (given a $\boldsymbol{b}$, just let $\boldsymbol{x} = D\boldsymbol{b}$; then $A\boldsymbol{x} = A(D\boldsymbol{b}) = I_m\boldsymbol{b} = \boldsymbol{b}$.
Therefore A has a pivot position in every row. This forces $m \leq n$, since every pivot position must be in a different column.

If $CA = I_n$, consider the equation $A\boldsymbol{x} = \mathbf{0}$. Then $CA\boldsymbol{x} = C\mathbf{0} = \mathbf{0}$. But $CA\boldsymbol{x} = I_n\boldsymbol{x} = \boldsymbol{x}$, so $\boldsymbol{x} = \mathbf{0}$. In other wirds, $A\boldsymbol{x} = \mathbf{0}$ has a unique solution and therefore the columns of A must be linearly independent and therefore each column must be a pivot column. Since each pivot position must be in a different row, this forces $n \leq m$.

So, combining the two paragraphs gives that $m = n$. Since A is now known to be square, the Invertible Matrix Theorem says that A is invertible and that $C = D = A^{-1}$.