True or false:

In a row reduction: if the original augmented matrix has no column of zeros, then the rref cannot have a column of zeros.

True. If you look at any nonzero column, you could use row replacements to turn all the nonzero elements in the column into)' except for one. When a column is reduced to having exactly one nonzero element, there ie no ERO that could be used to convert that last number into a 0.

Whenever a system has free variables, then the system has infinitely many solutions.

False: the system might be inconsistent. See the Existence and Uniqueness Theorem below (discussed in Lecture 2)

Existence and Uniqueness Theorem

Let A be the augmented matrix for a system of linear equations.

- The system is inconsistent if and only if

  a row with form $[0\ \ 0\ \ 0\ \ 0\ \ldots\ \ b]$, where $b \neq 0$, appears in an echelon form of A

  (*equivalently, if and only if the row* $[0\ \ 0\ \ 0\ \ldots\ 1]$ *appears in the row reduced echelon form of* A)

- Otherwise, the system is consistent, and either

  i) there is a unique solution (when there are no free variables)

  or

  ii) there are infinitely many solutions (when one or more variables are free)

What are the solutions of a system if the augmented matrix is

$$[1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6]$$

The system has one equation, 5 unknowns; the matrix is already in rref
One basic variable, x_1; all other variables are free.

Solutions (infinitely many) $x_1 = 6 - 2x_2 - 3x_3 - 4x_4 - 5x_5$
Parametric solution, expressing basic variable in terms of free variables (the "parameters")

What are the solutions of a system if the augmented matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$ (an echelon fom, but not rref)

Row reduce: $$\begin{bmatrix} 1 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & \mathbf{1} & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 \end{bmatrix}$$

Solutions (infinitely many)

$$\begin{cases} \boldsymbol{x_1} = 1 - x_2 \\ x_2 \text{ is free} \\ \boldsymbol{x_3} = 3 - x_4 \\ x_4 \text{ is free} \\ \boldsymbol{x_5} = 0 \end{cases}$$

What are the solutions of a system if the augmented matrix is

$$\begin{bmatrix} \mathbf{1} & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & \mathbf{1} & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

No solutions: 3rd row shows the system is inconsistent.

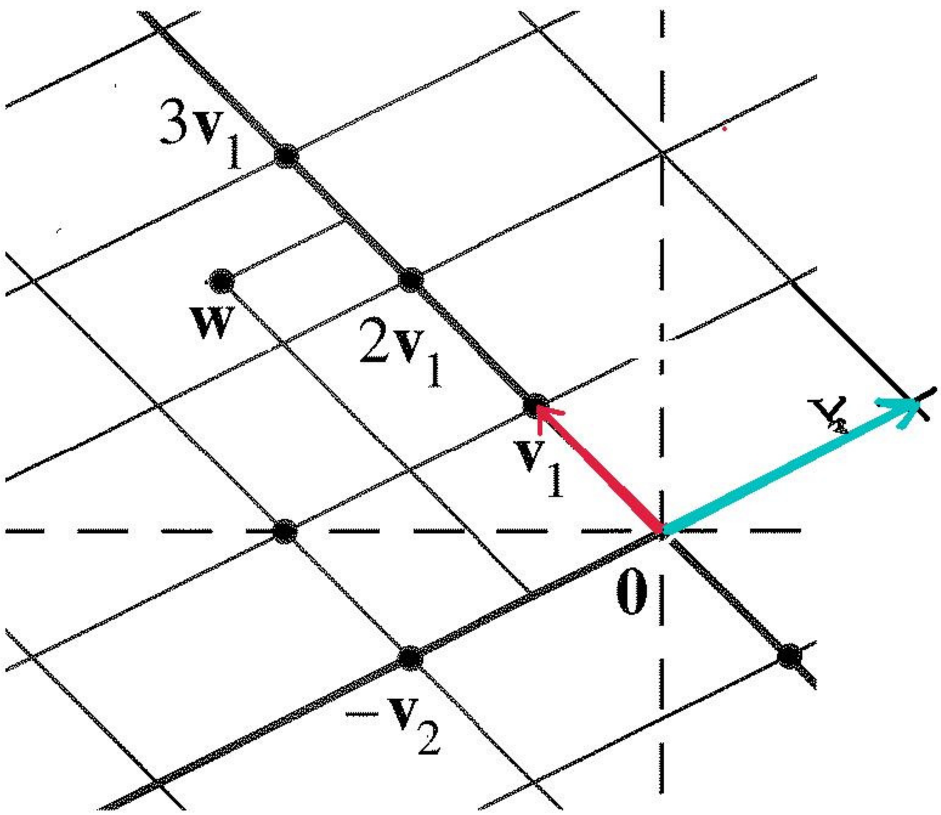
Properties of addition and scalar multiplication of vectors
(from textbook)

If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are vectors in $\mathbb{R}^n$ and c, d are scalars:

$\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{v}+\boldsymbol{u}$	$c(\boldsymbol{u}+\boldsymbol{v}) = c\boldsymbol{u}+c\boldsymbol{v}$
$(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w} = \boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$	$c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$
$\boldsymbol{u}+\boldsymbol{0} = \boldsymbol{0}+\boldsymbol{u} = \boldsymbol{u}$	$(c+d)\boldsymbol{u} = c\boldsymbol{u}+d\boldsymbol{u}$
$\boldsymbol{u}+(-\boldsymbol{u}) = -\boldsymbol{u}+\boldsymbol{u} = \boldsymbol{0}$	$1\boldsymbol{u} = \boldsymbol{u}$

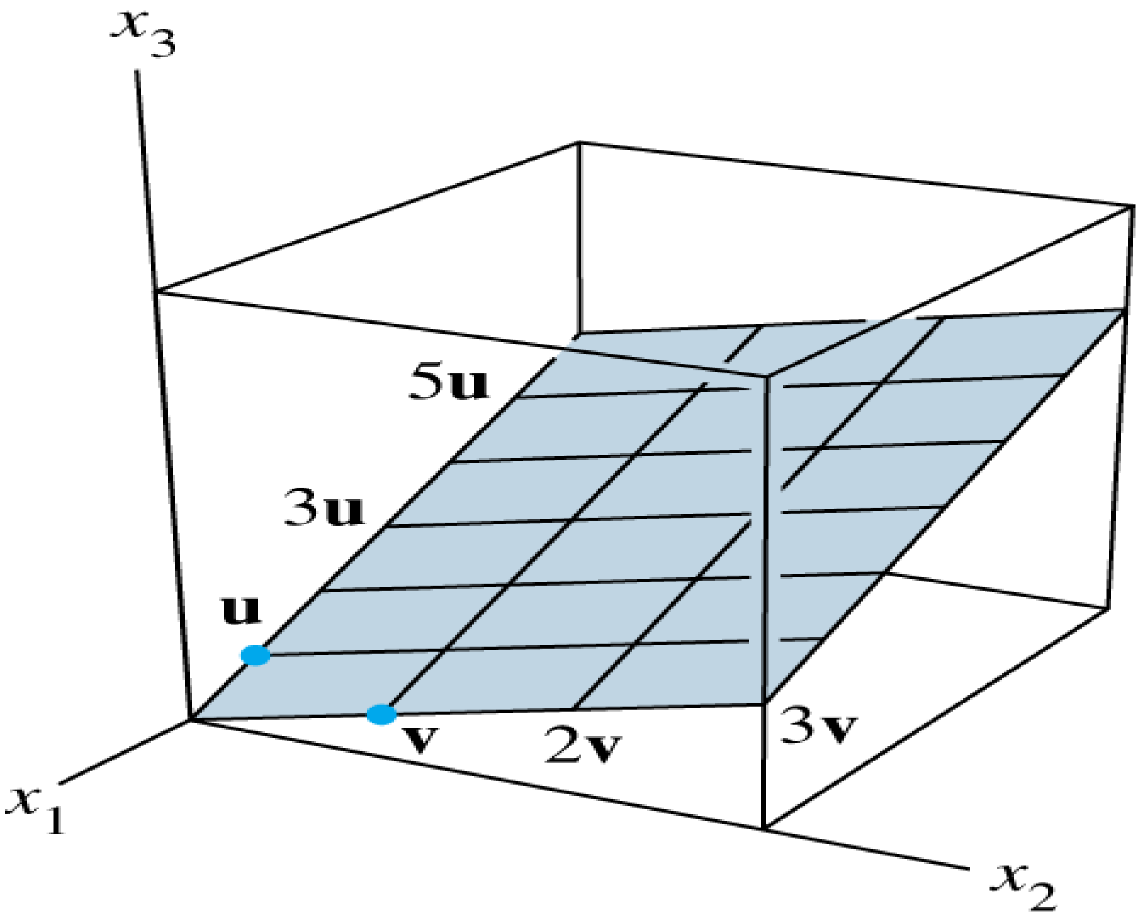
Every vector in $\mathbb{R}^2$ can be expressed as a linear combination of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. For example, it looks (approximately) like $\boldsymbol{w} = \frac{5}{2}\boldsymbol{v_1} - \frac{1}{2}\boldsymbol{v_2}$

In the figure, $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ can be used to create a "grid" of parallel lines: a new kind of "graph paper" where the "grid lines" are not perpendicular. On this "graph paper", $\boldsymbol{w}$ is located at $\left(\frac{5}{2}, -\frac{1}{2}\right)$



These two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^3$ lie in a <u>plane</u> in $\mathbb{R}^3$ (only a limited piece of which is pictured). Analogous to the preceding picture, $\boldsymbol{u}$ and $\boldsymbol{v}$ can be used to make a "coordinate grid" ("graph paper") on this <u>plane</u>.

The points <u>in this plane</u> are all the possible linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$;
So Span$\{\boldsymbol{u}, \boldsymbol{v}\}$ is the set of all points in this <u>plane</u>.



In particular, Span {u, v} <u>contains</u> the line in $\mathbb{R}^3$ through u and 0 and the line through v and 0. See the figure below.

Question: (in $\mathbb{R}^4$)

$$\text{Is} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix} \text{ in Span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -1 \\ 2 \end{bmatrix} \right\} ?$$

That is, does the **<u>vector equation</u>**

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix} + x \begin{bmatrix} -1 \\ 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ 3x_2 \\ x_1 - x_2 \\ 2x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix} \text{ have a solution?}$$

That is, does the **<u>system of linear equations</u>**

$$\begin{cases} 2x_1 - x_2 = 1 \\ 3x_2 = 2 \\ x_1 - x_2 = -1 \\ 2x_1 + 2x_2 = 4 \end{cases} \quad \text{have a solution?}$$

We solve the <u>vector equation</u> by solving the corresponding <u>linear system</u> (they are <u>equivalent)</u> and we do that by row reducing the augmented matrix

Augmented Matrix $=$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 2 \\ 1 & -1 & -1 \\ 2 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 2 \\ 2 & -1 & 1 \\ 2 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 4 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 3 & 2 \\ 0 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \\ 0 & 0 & -6 \end{bmatrix}$$

↙ **<u>INCONSISTENT</u>**

The system of linear equations has no solution, so the equivalent vector equation has no solution.

$$\text{That says: } \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix} \ \underline{\text{is not in}} \text{ Span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix} \right\}$$

In general

> $x_1 v_1 + \dots + x_p v_p = b$ (a <u>vector equation</u>)
>
> **has the same solutions as the <u>linear system</u> whose augmented matrix is**
>
> $[\,v_1 \; v_2 \;\; \dots \;\; v_p \mid b\,]$

<u>As illustrated above:</u>

$$\underset{}{x_1} \overset{v_1}{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}} + x \overset{v_2}{\begin{bmatrix} -1 \\ 3 \\ -1 \\ 2 \end{bmatrix}} = \begin{bmatrix} 2x_1 - x_2 \\ 3x_2 \\ x_1 - x_2 \\ 2x_1 + 2x_2 \end{bmatrix} = \overset{b}{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix}}$$

has the same solutions as $\begin{cases} 2x_1 - x_2 = 1 \\ 3x_2 = 2 \\ x_1 - x_2 = -1 \\ 2x_1 + 2x_2 = 4 \end{cases}$ for which the augmented matrix is

$$[\,v_1 \quad v_2 \quad b\,]$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 2 \\ 1 & -1 & -1 \\ 2 & 2 & 4 \end{bmatrix}$$