Homework set 10 - due Monday 12/02/24

Math 5047 – Renato Feres

For the remainder of the course we will use the text *Differential Geometry* by Loring W. Tu. The Olin library has the pdf for free download. This assignment is a work in progress. I expect to make changes which, however, should not (greatly) affect the exercises to be turned in.

Turn in problems 1(d), 2, 3(c) 5(b,c).

1. **Gauss-Bonnet for hypersurfaces.** Let M be an oriented hypersurface in $\mathbb{R}^{2m+1}$. (Thus M is a smooth manifold of even dimension $n = 2m$.) Let N be a unit normal vector field on M compatible with the orientation. Let $Sv := -D_v N$ be the shape operator of M, where D is the Euclidean connection in $\mathbb{R}^{n+1}$. Recall that S_p is a symmetric operator on T_pM at each $p \in M$. Its eigenvalues $k_1(p), \dots, k_n(p)$ are the *principal curvatures* of M at p. The *Gauss curvature* of M at p is defined as the product $K(p) = k_1(p) \dots k_n(p)$. Thus K is the determinant of the shape operator.

   The purpose of this exercise (mostly reading) is to prove (modulo the statement of the Poincaré-Hopft theorem) the Gauss-Bonnet theorem for hypersurfaces of even dimension n:

   $$\int_M K\omega = \frac{1}{2}\mathrm{Vol}\left(S^n\right)\chi(M). \tag{1}$$

   Here ω is the volume form on M.

   The *Gauss map* of the hypersurface is the map $g : M \to S^n$ defined by $g(p) = N(p)$, where the normal vector $N(p)$ can be regarded as a point in the unit sphere once we translate it to the origin of Euclidean space.

   Finally observe that the two affine subspaces T_pM and $T_{g(p)}S^n$ of $\mathbb{R}^{n+1}$ are translates of each other. Thus the differential $dg_p : T_pM \to T_{g(p)}S^n$ can be regarded as a linear map on the same vector space and its determinant makes sense.

   (a) Check that the Gauss curvature at p equals the determinant of the differential of g. (Note: with the identification of T_pM and $T_{g(p)}S^n$, the differential dg_p can be identified with the negative of the shape operator S_p. So the determinant of S_p and the determinant of dg_p differ by $(-1)^n$. But as n is even, the two determinants are the same.)

   (b) Let $\omega \in \Omega^n(M)$ denote the volume form on M and $\deg(g)$ the degree of the Gauss map. Let us recall here the definition of degree. Let $g : M \to S$ be a smooth map between compact, connected oriented manifolds of the same dimension n. Let $H^n(M)$ and $H^n(S)$ be the top degree de Rham cohomology groups. By definition $H^n(M) = \Omega^n(M)/d\Omega^{n-1}(M)$. It can be shown (using Stokes's theorem) that the map

   $$[\omega] \in H^n(M) \mapsto \int_M \omega \in \mathbb{R}$$

   is well-defined. It is also easy to check that this map is surjective. A deeper result (see, for example, *From Calculus to Cohomology: de Rham cohomology and characteristic classes* by Madsen and Tornehave) is that this map is an isomorphism. We then define the degree of g by the commutative diagram

$$\begin{array}{ccc} H^n(S) & \xrightarrow{g^*} & H^n(M) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{R} & \xrightarrow{\deg(g)} & \mathbb{R} \end{array}$$

Thus

$$\int_M g^*\omega = \deg(g)\int_S \omega.$$

The degree of g only depends on the homotopy class of g. A theorem from differential topology states the following. Choose a regular point p of the smooth map g and let $q_1,\dots,q_\ell$ be the inverse image of p. Let $\mathrm{Ind}(g,q_i)$ be either 1 or -1 depending on whether dg_{q_i} is orientation preserving or reversing. Then

$$\deg(g) = \sum_{q\in g^{-1}(p)} \mathrm{Ind}(g,q).$$

(See Theorem 11.9, page 101, of the above cited book.) In particular, the degree of g is an integer.

If we now take ω to denote the volume form on M, show that

$$\int_M K\omega = \deg(g)\mathrm{Vol}(S^n).$$

Note: by the change of variables in integration formula

$$\int_M K\omega = \int_M \det(dg_p)\omega = \int_M g^*\omega_S,$$

where ω_S is the volume form on S^n.

(c) In this and the next item, we show that the degree of the Gauss map g is half the Euler characteristic of M. From this observation we obtain the Gauss-Bonnet formula for hypersurfaces:

$$\int_M K\omega = \frac{1}{2}\mathrm{Vol}\left(S^n\right)\chi(M).$$

Let u be a point is S^n such that both u and $-u$ are regular values of g. (This is possible: if $\pi: S^n \to \mathbb{R}P^n$ is the projection map, pick a regular value of $\pi\circ g$.) Define the vector field $X \in \mathfrak{X}(M)$ such that $X(p)$ is the orthogonal projection of $-u$ to T_pM:

$$X(p) := -u + \left(u\cdot N(p)\right)N(p).$$

A point $p\in M$ is a singular point (zero) of X if and only if $u = (u\cdot N(p))N(p)$or, equivalently, $g(p) = \pm u$. In particular, u is perpendicular to T_pM. Since u and $-u$ are regular values of g and M is compact, X has only finitely many zeros. Observe that, at a zero p of X where $g(p) = \pm u$,

$$dX_p w = \pm\left(u\cdot dg_p w\right)u \pm dg_p w$$

for all $w\in T_pM$. But since g and N are really the same (up to translation) we have $dg_p w = D_w N \in T_pM$ so $u\cdot dg_p w = 0$. Therefore

$$dX_p = \begin{cases} +dg_p & \text{if } g(p) = u \\ -dg_p & \text{if } g(p) = -u. \end{cases}$$

In particular, dX_p maps T_pM into itself (at a zero point p). It follows that

$$\det\left(dX_p\right) = (\pm 1)^n \det\left(dg_p\right) = \det\left(dg_p\right)$$

since n is even. In class, we introduced the index of a vector field in dimension $n = 2$ in a somewhat informal way. A more precise definition (using the above notion of degree) shows that, if dX_p is an isomorphism (which is the case here since $g(p)$ is a regular value), then

$$\operatorname{Ind}(X, p) = \operatorname{sign}\left(\det\left(dX_p\right)\right).$$

We conclude that $\operatorname{Ind}(X, p) = +1$ if dg_p preserves orientation and -1 if dg_p reverses orientation. The index $\operatorname{Index}(X)$ is the sum of the indices of X at all zeros. Since the zeros of X are located at the inverse images under g of two regular values $\pm u$ we conclude that

$$\operatorname{Index}(X) = 2\deg(g).$$

We conclude that

$$\int_M K\omega = \frac{1}{2}\operatorname{Index}(X)\operatorname{Vol}(S^n).$$

But according to the Poincaré-Hopf theorem (which we discussed briefly in class), the index of X equals the Euler characteristic $\chi(M)$.

Our discussion of degree and index of vector fields has been very abbreviated. (This is a subject in differential topology that belongs to Geometry-Topology II.) For more on this topic and Gauss-Bonnet for hypersurfaces, see the classic *Differential Topology* by Victor Guillemin and Alan Pollack.

(d) Verify that Equation (1) holds when M is a sphere of radius R.

Solution.

(a) This is immediate since the determinant of the shape operator is the product of the principal curvatures.

(b) Nothing to do.

(c) Nothing to do.

◇

2. **The pfaffian of an even-dimensional hypersurface.** In Homework 9, Problem 8 we defined the pfaffian of a antisymmetric linear transformation of an even-dimensional vector space. Here we use it to define the Euler form and compute it for a hypersurface. You may want to refer to the solutions to that problem when answering the exercises enumerated below.

Let R be the curvature tensor of a connection ∇ on an oriented vector bundle $E \to M$ of even rank r. (We will shortly assume that $E = TM$.) We assume the connection is metric for a given Riemannian metric on E. If $\xi_1, \dots, \xi_r$ is a local, positive orthonormal frame of sections of E and $X, Y \in \mathfrak{X}(M)$, we define the following ordinary 2-forms on M:

$$\Omega_{ij}(X, Y) := \left\langle \xi_i, R(X, Y)\xi_j \right\rangle.$$

We thus obtain a matrix Ω with entries in $\Omega^2(M)$. This matrix is antisymmetric due to the symmetries of R. It thus makes sense to define a differential form $\operatorname{Pf}(\Omega) \in \Omega^r(M)$.

(a) Check that $\mathrm{Pf}(\Omega)$ does not depend on the choice of local orthonormal frame. Therefore it defines a global smooth form on M. Refer to item (c) of Problem 8 in Homework 9.

Note: we obtain directly from the above definition of Ω_{ij} the following change of frame equation. Let $\xi'_k = \sum_i F_{ki}\xi_i$ be a new local, positive orthonormal frame and $\Omega'_{k\ell}$ the curvature forms in this new frame. Then there is a smooth function (locally defined) F taking values in $SO(r)$, where r is the rank of E, so that $\Omega' = F\Omega F^\intercal$ holds.

(b) Let S denote the shape operator (page 235 of Lee's text) of a hypersurface $M \subset \mathbb{R}^{n+1}$. Let R be the curvature tensor of M. Show that

$$\langle R(W,X)Y,Z\rangle = \langle Z,SW\rangle\langle Y,SX\rangle - \langle Y,SW\rangle\langle Z,SX\rangle,$$

where $W,X,Y,Z \in \mathfrak{X}(M)$. (Use the Gauss Equation, Theorem 8.5, page 230, in Lee's text.)

(c) Let $e_1,\ldots,e_n$ be a local, positive orthonormal frame consisting of eigenvectors of S (recall that the shape operator is symmetric) with eigenvalues (principal curvatures) $k_1,\ldots,k_n$. Note that these k_i are smooth real-valued functions. Let $\theta_1,\ldots,\theta_n$ be the dual frame, so that $\theta_i(e_j) = \delta_{ij}$. Show that

$$\Omega_{ij} := \langle e_i, R(\cdot,\cdot)e_j\rangle = k_i k_j \theta_i \wedge \theta_j.$$

(d) Let $m = n/2$. Using the notation of Problem 8, Homework 9 we write

$$\alpha := \sum_{i<j} \Omega_{ij} e_i \wedge e_j = \frac{1}{2}\sum_{i,j} \Omega_{ij} e_i \wedge e_j.$$

By the definition of the pfaffian given in that assignment,

$$\alpha^m = \underbrace{\alpha \wedge \cdots \wedge \alpha}_{m} = m!\mathrm{Pf}(\Omega) e_1 \wedge \cdots \wedge e_{2m}.$$

Show that

$$\mathrm{Pf}(\Omega) = \frac{(2m)!}{2^m m!} K\omega$$

where $K = \prod k_i$ is the Gauss curvature and ω is the volume form.

Comparing the above expression for the pfaffian with the Gauss-Bonnet formula for hypersurfaces we see that the integrand in the latter formula is, in fact, the pfaffian of curvature.

3. **Induced connection on a pullback bundle.** (See section 22.10, page 210, of Loring Tu's text.) Let $\pi : E \to M$ be a smooth vector bundle of rank r over the manifold M. Let $f : N \to M$ be a smooth map and consider the pullback vector bundle f^*E. (This is defined in Section 20.4, page 177.) Given a connection ∇ on E, we wish to define a connection $\overline{\nabla}$ on f^*E that satisfies the following properties:

(a) If e is a smooth section of E and f^*e denotes the pullback section $(f^*e)(q) := \big(q, e(f(q))\big)$ then

$$\overline{\nabla}_u\left(f^*e\right) = \left(q, \nabla_{df_q u} e\right)$$

for any $u \in T_qN$. (Here $df_q u = (f_*)_q u : T_qN \to T_{f(q)}M$ is the standard tangent map.)

(b) The fundamental properties for a connection. (Definition 10.1, page 72.)

Show the following:

(a) The above properties uniquely specify a connection on f^*E.

(b) If $e_1, \dots, e_r$ is a local frame defined on an open set $U \subseteq M$ and ω_{ij} are the connection 1-forms for ∇ relative to this frame, then the pull-back $\overline{\omega}_{ij} := f^*\omega_{ij}$ are the connection 1-forms for $\overline{\nabla}$ relative to the local frame $f^*e_1, \dots, f^*e_r$ on $f^{-1}(U)$.

(c) With the same notation as in the previous item, let Ω_{ij} be the curvature 2-forms for ∇ relative to the local frame e_i. Then $\overline{\Omega}_{ij} = f^*\Omega_{ij}$ are the curvature 2-form for $\overline{\nabla}$ relative to the pullback frame.

(d) Let $\overline{R}$ denote the curvature tensor of $\overline{\nabla}$ and R the curvature tensor of ∇. Show that

$$\overline{R}_q(u,v)\overline{\xi} = \left(q, R_{f(q)}\left(df_q u, df_q v\right)\xi\right)$$

for $u, v \in T_qN$ and $\overline{\xi} = (q, \xi) \in (f^*E)_q$.

Further comments: We may consider the vector bundle $\Lambda^k(T^*M) \otimes E$ over M, whose sections are differential k-forms with coefficients in E. (See the next problem.) For such a form μ we define its pullback under f as

$$(f^*\mu)_q(v_1, \dots, v_k) = \left(q, \mu_{f(q)}\left(df_q v_1, \dots, df_q v_k\right)\right).$$

If $\mu = \alpha \otimes \xi$ is the tensor product of an ordinary differential form on M and a section of E, then the definition just given amounts to $f^*(\alpha \otimes \xi) = (f^*\alpha) \otimes (f^*\xi)$, where $f^*\alpha$ is the ordinary pullback of differential forms and $f^*\xi$ is as defined at the beginning of this exercise. If we think of the curvature tensor R as a section of $\Lambda^2(T^*M) \otimes \mathrm{End}(E)$, then under this definition $\overline{R} = f^*R$ and $\overline{R}_q(v_1, v_2)\overline{\xi}_q = (f^*R)_q(v_1, v_2)(f^*\xi)_q$. Also note that the definition of $\overline{\nabla}$ implies $\overline{\nabla}(f^*\xi) = f^*(\nabla\xi)$.

Solution.

(a) Let ξ be a section of f^*E and $e_1, \dots, e_r$ a local frame of E on the open set $U \subset M$. On $f^{-1}(U)$, we may express ξ in terms of the pullback frame $\overline{e}_i := f^*e_i$: $\xi = \sum_i h_i \overline{e}_i$. Then a connection $\overline{\nabla}$ having the desired properties must satisfy:

$$\overline{\nabla}_u \xi = \sum_i \left((uh_i)\overline{e}_i(q) + h_i(q)\overline{\nabla}_u \overline{e}_i\right) = \sum_i \left((uh_i)\overline{e}_i + h_i(q)\left(q, \nabla_{df_q u} e_i\right)\right)$$

for all $u \in T_qN$ and $q \in f^{-1}(U)$. This implies uniqueness. We need to check that this is well-defined (i.e., independent of the local frame) and is indeed a connection.

To verify that $\overline{\nabla}$ is well-defined, let $\xi_1, \dots, \xi_r$ be another choice of local frame on U. Let the change of frame matrix-valued function be denoted $A = \left(a_{ij}\right) : U \to GL(r, \mathbb{R})$, so that $\xi_j = \sum_i a_{ij} e_j$. The change of frame matrix-valued function on $f^{-1}(M)$ for the pullback frames is then $\overline{A} := A \circ f = \left(a_{ij} \circ f\right)$.

A section ξ of f^*E can then be expressed in both frames at any $q \in f^{-1}(U)$ as:

$$\xi(q) = \sum_j g_j(q)\overline{\xi}_j(q) = \sum_{i,j} g_j(p)\overline{a}_{ij}(q)\overline{e}_i(q).$$

We observe that

$$\begin{aligned}
\overline{\nabla}_u \sum_j g_j \overline{\xi}_j &= \overline{\nabla}_u \sum_{i,j} g_j \overline{a}_{ij} \overline{e}_i \\
&= \sum_{i,j} \left(u\left(g_j \overline{a}_{ij}\right)\overline{e}_i(q) + g_j(q)\overline{a}_{ij}(q)\overline{\nabla}_u \overline{e}_i\right) \\
&= \sum_{i,j} (ug_j)\overline{a}_{ij}(q)\overline{e}_i(q) + \sum_{i,j} g_j(q)\left[\left(u\overline{a}_{ij}\right)\overline{e}_i(q) + \overline{a}_{ij}(q)\overline{\nabla}_u \overline{e}_i\right] \\
&= \sum_j \left[(ug_j)\overline{\xi}_j(q) + g_j(q)\overline{\nabla}_u \overline{\xi}_j\right].
\end{aligned}$$

Thus we have

$$\sum_i \left((uh_i)\overline{e}_i(q) + h_i(q)\overline{\nabla}_u \overline{e}_i\right) = \sum_j \left((ug_j)\overline{\xi}_j(q) + g_j(q)\overline{\nabla}_u \overline{\xi}_j\right).$$

Therefore the definition does not depend on the choice of local pullback frame.

It is clear from the expression for $\overline{\nabla}$ given in terms of a pullback frame that $\overline{\nabla}_u \xi$ is $C^\infty(M)$-linear (linear over functions) in u is $\mathbb{R}$-linear in ξ. The Leibniz property also holds:

$$\begin{aligned}\nabla_u(g\xi) &= \sum_i \left(u(h_i g)\overline{e}_i(q) + h_i(q)g(q)\overline{\nabla}_u \overline{e}_i\right) \\ &= \sum_i (ug)h_i(q)\overline{e}_i(q) + g(q)\sum_i \left((uh_i)\overline{e}_i(q) + h_i(q)\overline{\nabla}_u \overline{e}_i\right) \\ &= (ug)\xi(q) + g(q)\nabla_u \xi.\end{aligned}$$

(b) Let $\overline{\omega}_{ij}$ be the connection 1-forms for $\overline{\nabla}$ relative to the pullback frame f^*e_i. Then

$$\sum_i \overline{\omega}_{ij}(u)\overline{e}_i(q) = \overline{\nabla}_u \overline{e}_j = \left(q, \nabla_{df_q u} e_j\right) = \left(q, \sum_i \omega_{ij}(df_q u)e_i(f(q))\right) = \sum_i \left(f^*\omega_{ij}\right)(u)\overline{e}_i(q).$$

We conclude that $\overline{\omega}_{ij} = f^*\omega_{ij}$.

(c) (HW)

(d) It is enough to check the relation for $\overline{\xi} = \overline{E}_j$. In this case

$$\overline{R}_q(u,v)\overline{e}_j(q) = \sum_i \overline{\Omega}_{ij}(u,v)\overline{e}_i(q) = \left(q, \sum_i \Omega_{ij}(df_q u, df_q v)e_i(f(q))\right) = \left(q, R_{f(q)}(df_q u, df_q v)e_j(f(q))\right).$$

⋄

4. **More on vector bundle-valued differential forms.** Let $\pi : E \to M$ be a smooth vector bundle over the manifold M. Recall that differential k-forms on M with coefficients in E are defined as sections of the vector bundle $\Lambda^k(T^*M) \otimes E$. The space of such sections will be denoted $\Omega^k(M,E) := \Gamma\left(\Lambda^k(T^*M)\otimes E\right)$. In this definition, E may be replaced with other vector bundles obtained from E. We are especially interested in the bundle $\Lambda^*(E^*)$, that is, the exterior algebra bundle of alternating forms on E. This is the vector bundle over M whose fiber over $p \in M$ is the exterior algebra $\Lambda^*\left(E_p^*\right) = \bigoplus_k \Lambda^k\left(E_p^*\right)$, where E_p^* is the dual vector space to E_p. An element

$$\mu \in \Omega^k\left(M, \Lambda^\ell(E^*)\right) := \Gamma\left(\Lambda^k(T^*M)\otimes\Lambda^\ell(E^*)\right)$$

will be called a (k,ℓ)-differential form. Note that at any $p \in M$ and given $v_1,\dots,v_k \in T_pM$,

$$\mu_p(v_1,\dots,v_k) \in \Lambda^\ell\left(E_p^*\right).$$

Thus it makes sense to write $\mu_p(v_1,\dots,v_k)(u_1,\dots,u_\ell) \in \mathbb{R}$ (or $\mathbb{C}$ if E is complex), where the u_i are elements of E_p, and the resulting number is anti-symmetric separately in the v_i and in the u_j arguments.

For example, if E is Riemannian, with metric $\langle\cdot,\cdot\rangle$ and ∇ is a metric connection on E, then

$$\mathbf{R}_p(v_1,v_2)(u_1,u_2) := \langle u_1, R(v_1,v_2)u_2\rangle_p$$

defines a section $\mathbf{R} \in \Omega^2\left(M, \Lambda^2(E^*)\right)$, where R is the curvature tensor of ∇. Given a local orthonormal frame

$e_1, \dots, e_r$ of E, where r is the rank of the vector bundle, we have already employed the notation

$$\Omega_{ij}(v_1, v_2) := \mathbf{R}(v_1, v_2)(e_i, e_j).$$

This is a locally defined ordinary 2-form on M.

If μ is a differential (r, j)-form on M and ν is a differential (s, k)-form, we define their wedge product as the $(r+s, j+k)$-form $\mu \wedge \nu$ as follows:

$$(\mu \wedge \nu)(v_1, \dots, v_{r+s}) := \frac{1}{r!s!} \sum_{\sigma \in S_{r+s}} \mathrm{sign}(\sigma) \mu(v_{\sigma(1)}, \dots, v_{\sigma(r)}) \wedge \nu(v_{\sigma(r+1)}, \dots, v_{\sigma(r+s)}).$$

Notice the need for the wedge product symbol on the right-hand side of this identity. This is the wedge product on the exterior $\Lambda^*(E^*)$.

(a) Show that if μ is a (r, i)-form and ν is an (s, j)-form, then

$$\mu \wedge \nu = (-1)^{rs+ij} \nu \wedge \mu.$$

(b) Show that if $\mu = \alpha \otimes \xi$ and $\nu = \beta \otimes \eta$, then $\mu \wedge \nu = (\alpha \wedge \beta) \otimes (\xi \wedge \eta)$. Here α, β are ordinary forms on M and ξ, η are sections of $\Lambda^*(E^*)$.

(c) Suppose E is an oriented Riemannian vector bundle over M of rank $2m$. (The even parity of the rank is not required in this part of the exercise, but will be needed later.) Let $e_1, \dots, e_{2m}$ be a local positive (relative to the given orientation) orthonormal frame for E over an open set $U \subset M$. Let $\epsilon_1, \dots, \epsilon_{2m}$ be the dual frame, so that $\epsilon_i = \langle e_i, \cdot \rangle$. Then $\omega = \epsilon_1 \wedge \dots \wedge \epsilon_{2m}$ is the Riemannian volume form on E.

   i. Argue that ω is globally defined.

   ii. Show that ω is parallel: $\nabla \omega = 0$.

(d) We define the Chern-Euler form $\mathbf{e} \in \Omega^{2m}\left(M, \Lambda^{2m}(E^*)\right)$ as

$$\mathbf{e} = \frac{\mathbf{R}^m}{m!(2\pi)^m},$$

where $\mathbf{R}$ is the curvature form defined above. (Note: this is a frame-independent version of the curvature matrix previously defined.) I am writing the exterior power $\mathbf{R} \wedge \dots \wedge \mathbf{R}$ as $\mathbf{R}^m$.

Show that

$$\mathbf{e} = \mathrm{Pf}\left(\frac{\Omega}{2\pi}\right) \otimes \omega.$$

Here Ω is the curvature matrix relative to a choice of positive orthonormal frame, whose entries are:

$$\Omega_{ij}(v_1, v_2) = \Omega(v_1, v_2)(e_i, e_j).$$

In other words,

$$\Omega(v_1, v_2) = \frac{1}{2} \sum_{i,j} \Omega_{ij}(v_1, v_2) \epsilon_i \wedge \epsilon_j = \sum_{i<j} \Omega_{ij}(v_1, v_2) \epsilon_i \wedge \epsilon_j.$$

(Suggestion: use the definition of the Pfaffian given in Exercise 1 of Homework set 9.)

Note: the form ω trivializes $\Lambda^{2m}(E^*)$. This allows us to identity $\mathbf{e} = \mathrm{Pf}(\Omega/2\pi)$, which we do going forward.

Solution.

(a) Let η be the permutation

$$\eta(1)=r+1,\ \eta(2)=r+2,\ \ldots,\ \eta(s)=r+2,\ \eta(1+s)=1,\ \eta(2+s)=2,\ \ldots,\ \eta(r+s)=r,$$

which has sign $\operatorname{sign}(\eta)=(-1)^{rs}$. First note that

$$\begin{aligned}(\mu\wedge\nu)(v_1,\ldots,v_{r+s}) &:= \frac{1}{r!s!}\sum_{\sigma\in S_{r+s}}\operatorname{sign}(\sigma)\mu(v_{\sigma(1)},\ldots,v_{\sigma(r)})\wedge\nu(v_{\sigma(r+1)},\ldots,v_{\sigma(r+s)})\\ &= (-1)^{ij}\frac{1}{r!s!}\sum_{\sigma\in S_{r+s}}\operatorname{sign}(\sigma)\nu(v_{\sigma(r+1)},\ldots,v_{\sigma(r+s)})\wedge\mu(v_{\sigma(1)},\ldots,v_{\sigma(r)})\\ &= (-1)^{ij}\frac{1}{r!s!}\sum_{\sigma\in S_{r+s}}\operatorname{sign}(\sigma)\nu(v_{\sigma(\eta(1))},\ldots,v_{\sigma(\eta(s))})\wedge\mu(v_{\sigma(\eta(1+s))},\ldots,v_{\sigma(\eta(r+s))})\\ &= (-1)^{ij+rs}\frac{1}{r!s!}\sum_{\sigma\in S_{r+s}}\operatorname{sign}(\sigma)\nu(v_{\sigma(1)},\ldots,v_{\sigma(s)})\wedge\mu(v_{\sigma(r+1)},\ldots,v_{\sigma(r+s)})\\ &= (-1)^{ij+rs}(\nu\wedge\mu)(v_1,\ldots,v_{r+s})\end{aligned}$$

(b) We may assume that μ is an (r,i)-form and ν is an (s,j)-form. Then

$$\begin{aligned}(\mu\wedge\nu)(v_1,\ldots,v_{r+s}) &= \frac{1}{r!s!}\sum_{\sigma\in S_{r+s}}\operatorname{sign}(\sigma)\mu(v_{\sigma(1)},\ldots,v_{\sigma(r)})\wedge\nu(v_{\sigma(r+1)},\ldots,v_{\sigma(r+s)})\\ &= \frac{1}{r!s!}\sum_{\sigma\in S_{r+s}}\operatorname{sign}(\sigma)\left(\alpha(v_{\sigma(1)},\ldots,v_{\sigma(r)})\xi\right)\wedge\left(\beta(v_{\sigma(r+1)},\ldots,v_{\sigma(r+s)})\eta\right)\\ &= \frac{1}{r!s!}\sum_{\sigma\in S_{r+s}}\operatorname{sign}(\sigma)\alpha(v_{\sigma(1)},\ldots,v_{\sigma(r)})\beta(v_{\sigma(r+1)},\ldots,v_{\sigma(r+s)})\xi\wedge\eta\\ &= (\alpha\wedge\beta)(v_1,\ldots,v_{r+s})\xi\wedge\eta\\ &= \left((\alpha\wedge\beta)\otimes(\xi\wedge\eta)\right)(v_1,\ldots,v_{r+s}).\end{aligned}$$

We conclude that $\mu\wedge\nu=(\alpha\wedge\beta)\otimes(\xi\wedge\eta)$.

(c) Let ω be the Riemannian volume form on E, as defined locally above.

i. If $\overline{e}_1,\ldots,\overline{e}_{2m}$ is another local, positive orthonormal frame on some neighborhood $\overline{U}$ that overlaps with U, then on their intersection the change of frame matrix A lies in $SO(2m)$. It follows that

$$e_1\wedge\cdots\wedge e_{2m}=\det(A)\overline{e}_1\wedge\cdots\wedge\overline{e}_{2m}=\overline{e}_1\wedge\cdots\wedge\overline{e}_{2m}.$$

Thus ω does not depend on the choice of the positive orthonormal frame. Therefore it defines a global form.

ii. Let $v\in T_pM$ for $p\in M$ and $\gamma(t)$ a smooth curve satisfying $\gamma(0)=p$ and $\gamma'(0)=v$. Let $e_1(t),\ldots,e_{2m}(t)$ be parallel sections of E along γ, so that $\frac{\nabla e_i}{dt}(0)=0$. Then

$$\nabla_v\omega=\left.\frac{\nabla}{dt}\right|_{t=0}\epsilon_1(t)\wedge\cdots\wedge\epsilon_{2m}(t)=\sum_{i=1}^{2m}\epsilon_1\wedge\cdots\wedge\left.\frac{\nabla\epsilon_i}{dt}\right|_{t=0}\wedge\cdots\wedge\epsilon_{2m}.$$

But if the $e_i(t)$ are parallel along γ, then so are the $\epsilon_i(t)$. This follows from the fact that the connection is metric. Therefore $\nabla_v\omega=0$.

(d) As noted, we can write $\Omega=\frac{1}{2}\sum_{ij}\Omega_{ij}\otimes\epsilon_i\wedge\epsilon_j$. Using the definition of the Pfaffian from Homework 9 (1), we readily obtain

$$\mathbf{R}^m = m!\mathrm{Pf}(\Omega) \otimes \epsilon_1 \wedge \cdots \wedge \epsilon_{2m} = m!\mathrm{Pf}(\Omega) \otimes \omega. \tag{2}$$

The expression for the Chern-Euler form in terms of the Pfaffian follows from this identity.

Note 1: We have already seen (homework 9) that $\mathrm{Pf}(\Omega)$ does not depend on the choice of positive local orthonormal frame e_i, so this differential $2m$-form is globally defined on M. Equation 2 makes this explicit.

Note 2: We will see in the next problem that $d^\nabla \mathbf{R} = 0$, which is an expression of the second Bianchi identity.

Note 3: It can be shown that the cohomology class of the Chern-Euler form $\mathbf{e} = \mathrm{Pf}(\Omega/2\pi)$ is independent of the Riemannian metric on M and the connection ∇ on E. It does depend, however, on the orientation of E. It is called the *Euler class* of the oriented vector bundle E.

Note 4: The Euler class is *natural*: if f maps N to M, then the Euler class of the pull-back bundle f^*E is the pullback by f of the Euler class of E. (See the previous Exercise 3.) In addition, the Euler class of the Whitney sum $E_1 \oplus E_2$ of oriented vector bundles of M is the product of the Euler classes of E_1 and E_2. Finally, the Euler class of E is *integral*: if the rank of E is $2k$ then the Euler class of E is in $H^{2k}(M;\mathbb{Z})$.

Note 5: The general Gauss-Bonnet theorem states:

Theorem 0.1 (Gauss-Bonnet-Allendoerfer-Weil-Chern)**.** *Let M be a $2m$-dimensional, oriented, compact Riemannian manifold (without boundary). Then*

$$\int_M \mathrm{Pf}\left(\frac{\Omega}{2\pi}\right) = \chi(M).$$

Note 6: What follows is further remarks towards the proof of this theorem. With this in mind, you may take throughout $E = TM$, although the more general case is useful.

⋄

5. **More on the exterior covariant derivative.** Different from Homework 9, I'll define here the exterior covariant derivative for a vector bundle E as the map

$$d^\nabla : \Omega^k(M,E) \to \Omega^{k+1}(M,E)$$

such that, at a point $p \in M$,

$$\left(d^\nabla \psi\right)(v_0,\ldots,v_k) = \sum_i (-1)^i \nabla_{v_i} \psi(V_0,\ldots,\hat{V}_i,\ldots,V_k) + \sum_{i<j} \psi\left([V_i,V_j], v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_k\right) \tag{3}$$

for $\psi \in \Omega^k(M,E)$, $v_0,\ldots,v_k \in T_pM$, and $V_0,\ldots,V_k \in \mathfrak{X}(M)$ such that $V_i(p) = v_i$. It follows from this definition that if ξ is a section of E, hence an element of $\Omega^0(M,E) = \Gamma(E)$, then $d^\nabla \xi = \nabla\xi$; and for $\omega \otimes \xi \in \Omega^k(M,E)$, where ω is an ordinary k-form on M and ξ a section of E,

$$d^\nabla(\omega \otimes \xi) = d\omega \otimes \xi + (-1)^k \omega \wedge d^\nabla \xi. \tag{4}$$

As before in this assignment, we are especially interested here in the case where the vector bundle in question is the exterior algebra $\Lambda^*(E^*)$ over a given vector bundle E. Keep in mind that if ξ, η are sections of $\Lambda^i(E^*)$ and

$\Lambda^j(E^*)$, respectively, and $v \in T_pM$, then

$$\nabla_v(\xi \wedge \eta) = (\nabla_v \xi) \wedge \eta + \xi \wedge \left(\nabla_v \eta\right).$$

(The definition in Section 22.5, Proposition 22.7, page 206, in Loring Tu's textbook extends to tensors over a general vector bundle E, not only TM.) Note that, different from expressions involving the exterior derivative, there is no sign in this Leibniz rule for the covariant derivative.

(a) Show that the definition of d^∇ given by Equation (3) implies the identity (4).

(b) If μ is an (r,i)-form and ν is (s,j)-form, show that

$$d^\nabla(\mu \wedge \nu) = \left(d^\nabla \mu\right) \wedge \nu + (-1)^r \mu \wedge d^\nabla \nu.$$

(c) If $f : N \to M$ is a smooth map and $E \to M$ is a smooth vector bundle over M, let $\overline{\nabla}$ be the pullback connection on f^*E. Show that $f^* \circ d^\nabla = d^{\overline{\nabla}} \circ f^*$.

(d) Let $\mathbf{R}$ be the curvature $(2,2)$-form defined in the previous exercise for a given metric connection ∇ on the Riemannian vector bundle E. Show that $d^\nabla \mathbf{R} = 0$. This is an expression of the Second Bianchi Identity. Also show that $d^\nabla R = 0$. The latter identity does not depend on the connection being metric and on E being Riemannian.

(e) Recalling from the previous exercise the relation between Ω^m and the Pfaffian, show that if E has rank $2m$, then $\mathrm{Pf}(\Omega)$ is a closed $2m$-form (of the standard kind) on M.

Solution.

(a) Identity (4) can be seen as follows. Let X_0,

$$\begin{aligned}
d^\nabla(\omega \otimes \xi)(v_0, \ldots, v_k) &= \sum_i (-1)^i \nabla_{v_i}\left(\omega(V_0, \ldots, \hat{V}_i, \ldots, V_k)\xi\right) + \sum_{i<j} \psi\left([V_i, V_j], v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_k\right)\xi \\
&= \left(\sum_i (-1)^i v_i \omega(V_0, \ldots, \hat{V}_i, \ldots, V_k) + \sum_{i<j} \psi\left([V_i, V_j], v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_k\right)\right)\xi \\
&\qquad + \sum_i (-1)^i \omega(v_0, \ldots, \hat{v}_i, \ldots, v_k) \nabla_{v_i}\xi \\
&= d\omega(v_0, \ldots, v_k)\xi + \sum_i (-1)^i \omega(v_0, \ldots, \hat{v}_i, \ldots, v_k)\nabla_{v_i}\xi.
\end{aligned}$$

It is helpful here to use the shuffle form of the wedge product (Proposition 19.15, page 172 of Tu's textbook). Let π_i be the shuffle permutation given in cycle form as $\pi_i = (i\ i+1\ \cdots\ k)$. Note that $\mathrm{sign}(\pi_i) = (-1)^{k+i}$ and

$$\left(v_{\pi_i(0)}, \ldots, v_{\pi_i(k)}\right) = (v_0, \ldots, \hat{v}_i, \ldots, v_k, v_i).$$

It follows that

$$\left(\omega \wedge d^\nabla \xi\right)(v_0, \ldots, v_k) = \sum_i (-1)^{k+i} \omega(v_0, \ldots, \hat{v}_i, \ldots, v_k)\nabla_{v_i}\xi.$$

Therefore $d^\nabla(\omega \otimes \xi) = d\omega \otimes \xi + (-1)^k \omega \wedge d^\nabla \xi$.

(b) (HW)

(c) (HW)

(d) Let V_0, V_1, V_2 be vector fields on M and ξ_0, ξ_1 sections on E. We wish to evaluate $\left(d^\nabla \mathbf{R}\right)(V_0, V_1, V_2)(\xi_0, \xi_1)$. From the definition of d^∇ we have

$$\left(d^\nabla \mathbf{R}\right)(V_0,V_1,V_2) = \nabla_{V_0}\mathbf{R}(V_1,V_2) - \nabla_{V_1}\mathbf{R}(V_0,V_2) + \nabla_{V_2}\mathbf{R}(V_0,V_1) - \mathbf{R}([V_0,V_1],V_2) + \mathbf{R}([V_0,V_2],V_1) - \mathbf{R}([V_1,V_2],V_0).$$

And from the general properties of ∇ acting on tensors over E,

$$\begin{aligned}\left(\nabla_{V_0}\mathbf{R}(V_1,V_2)\right)(\xi_0,\xi_1) &= V_0\mathbf{R}(V_1,V_2)(\xi_0,\xi_1) - \mathbf{R}(V_1,V_2)\left(\nabla_{V_0}\xi_0,\xi_1\right) - \mathbf{R}(V_1,V_2)\left(\xi_0,\nabla_{V_0}\xi_1\right)\\ &= V_0\langle\xi_0, R(V_1,V_2)\xi_1\rangle - \left\langle\nabla_{V_0}\xi_0, R(V_1,V_2)\xi_1\right\rangle - \left\langle\xi_0, R(V_1,V_2)\nabla_{V_0}\xi_1\right\rangle\\ &= \left\langle\xi_0, \nabla_{V_0}R(V_1,V_2)\xi_1 - R(V_1,V_2)\nabla_{V_0}\xi_1\right\rangle\\ &= \left\langle\xi_0, \left(\nabla_{V_0}R(V_1,V_2)\right)\xi_1\right\rangle.\end{aligned}$$

Thus

$$\begin{aligned}\left(d^\nabla \mathbf{R}\right)(V_0,V_1,V_2)(\xi_0,\xi_1) &= \left\langle\xi_0, \left(\nabla_{V_0}R(V_1,V_2)\right)\xi_1\right\rangle - \left\langle\xi_0, \left(\nabla_{V_1}R(V_0,V_2)\right)\xi_1\right\rangle + \left\langle\xi_0, \left(\nabla_{V_2}R(V_0,V_1)\right)\xi_1\right\rangle\\ &\qquad - \langle\xi_0, R([V_0,V_1],V_2)\xi_1\rangle + \langle\xi_0, R([V_0,V_2],V_1)\xi_1\rangle - \langle\xi_0, R([V_1,V_2],V_0)\xi_1\rangle\\ &= \left\langle\xi_0, \left(d^\nabla R\right)(V_0,V_1,V_2)\xi_1\right\rangle.\end{aligned}$$

Thus it suffices to show $d^\nabla R = 0$. To simplify expanding the terms of $\left(d^\nabla R\right)(V_0,V_1,V_2)\xi$ I will use the following notation: $\nabla_{V_i} = |_i$, $R(V_i,V_j) = (i,j)$, $[V_i,V_j] = [i,j]$. Thus, for example, we write

$$\nabla_{V_i}R(V_j,V_k)\xi = |_i(j,k),\quad R(V_j,V_k)\nabla_{V_0}\xi = (j,k)|_i,\quad R([V_i,V_j],V_k)\xi = ([i,j],k),\quad \nabla_{[V_i,V_j]}\nabla_{V_k}\xi = |_{[i,j]}|_k$$

and so on. We can then write

$$\begin{aligned}\left(d^\nabla R\right)(V_0,V_1,V_2)\xi &= |_0(1,2) - (1,2)|_0 - |_1(0,2) + (0,2)|_1 + |_2(0,1) - (0,1)|_2 - ([0,1],2) + ([0,2],1) - ([1,2],0)\\ &= |_0\left(|_1|_2 - |_2|_1 - |_{[1,2]}\right) - \left(|_1|_2 - |_2|_1 - |_{[1,2]}\right)|_0 - |_1\left(|_0|_2 - |_2|_0 - |_{[0,2]}\right) + \left(|_0|_2 - |_2|_0 - |_{[0,2]}\right)|_1\\ &\quad + |_2\left(|_0|_1 - |_1|_0 - |_{[0,1]}\right) - \left(|_0|_1 - |_1|_0 - |_{[0,1]}\right)|_2 - \left(|_{[0,1]}|_2 - |_2|_{[0,1]} - [[0,1],2]\right)\\ &\quad + \left(|_{[0,2]}|_1 - |_1|_{[0,2]} - [[0,2],1]\right) - \left(|_{[1,2]}|_0 - |_0|_{[1,2]} - [[1,2],0]\right)\\ &= [[0,1],2] - [[0,2],1] + [[1,2],0]\\ &= 0.\end{aligned}$$

The Jacobi identity was used at the last step.

(e) Since $d^\nabla\mathbf{R} = 0$ and $d^\nabla\mathbf{R}\wedge\Theta = (d^\nabla\mathbf{R})\wedge\Theta + \mathbf{R}\wedge\Theta$, a simple induction shows that $d^\nabla\mathbf{R}^m = 0$. So

$$0 = (m!)^{-1}d^\nabla\mathbf{R}^m = d^\nabla\left(\mathrm{Pf}(\Omega)\otimes\omega\right) = (d\mathrm{Pf}(\Omega))\otimes\omega + (-1)^{2m}\mathrm{Pf}((\Omega))\wedge\nabla\omega.$$

Here ω is the Riemannian volume form of the oriented Riemannian bundle E of rank $2m$. By Exercise 4, ω is parallel. Therefore $(d\mathrm{Pf}(\Omega))\otimes\omega = 0$. As ω is nowhere vanishing, we conclude that $d\mathrm{Pf}(\Omega) = 0$.
For the application to Gauss-Bonnet-Chern, this is trivial since $\mathrm{Pf}(\Omega)$ is a top degree form.

⋄

6. **The canonical section and canonical form on π^*E.** Using the bundle map $\pi : E \to M$ we can pull back E under π. Then π^*E is a vector bundle over the manifold E. By the definition of pullback bundles, elements of π^*E have

the form $(e,e') \in E \times E$ such that $\pi(e) = \pi(e')$. Another vector bundle we can define over the manifold E is the *vertical bundle* V. The vector fiber at each $e \in E$ is the kernel of $d\pi_e$: $V_e = \{\xi \in T_eE : d\pi_e\xi = 0\}$. Note that V_e is linearly isomorphic to $E_{\pi(e)}$, with the isomorphism given by the map

$$\mathcal{I}_e : e' \in E_{\pi(e)} \mapsto \left.\frac{d}{dt}\right|_{t=0} \left(e + te'\right) \in V_e.$$

Given a connection ∇ on E, we can define the *horizontal* subbundle of TE, denoted H. The fiber H_e is the kernel of a map $\mathcal{K}_e : T_eE \to E_{\pi(e)}$ which we define as follows. An element $\xi \in T_eE$ can be represented by a curve $e(t) \in E$, so that $e(0) = e$ and $e'(0) = \xi$. Then $e(t)$ is a section of E over the curve $\gamma(t) = \pi(e(t)) \in M$. Therefore it makes sense to define

$$\mathcal{K}_e\xi = \left.\frac{\nabla e}{dt}\right|_{t=0} \in E_{\pi(e)}.$$

Note that $\mathcal{K}_e \circ \mathcal{I}_e$ is the identity map on $E_{\pi(e)}$. Moreover, the rank of H is equal to the dimension of M and $d\pi_e : H_e \to T_{\pi(e)}M$ and $\mathcal{K}_e : V_e \to E_{\pi(e)}$ are linear isomorphisms for each e, as can be easily checked. Also observe that $\mathcal{I}_e \circ \mathcal{K}_e$ is the identity on V_e. From these definitions we obtain a splitting of TE as a direct sum of subbundles: $TE = V \oplus H$.

The pullback bundle π^*E admits a canonical section $\mathscr{s} \in \Gamma(\pi^*E)$ such that $\mathscr{s}(e) = (e,e)$. Given the isomorphism of π^*E and V that you will establish below, $\mathscr{s}$ may be regarded as a section of V, in which case it is given by $\mathscr{s}(e) = \mathcal{I}_e e$.

Further define the $(0,1)$-form η on π^*E (which we may regard as a section of the dual vertical bundle V^* over E):

$$\eta_e\left(\mathcal{I}_e e'\right) = \left\langle \mathscr{s}(e), \mathcal{I}_e e'\right\rangle_e = \left\langle e, e'\right\rangle_{\pi(e)}.$$

We are using here the pullback metric to π^*E, also denoted by $\langle\cdot,\cdot\rangle$. I note that $\overline{\nabla}$ is a metric connection for this metric.

Finally, we define the *connection* $(1,1)$-form $\theta = d^{\overline{\nabla}}\eta \in \Gamma(T^*E \otimes V^*)$.

(a) Show that π^*E and V are isomorphic vector bundles over E.

(b) If $\overline{\nabla}$ is the pullback connection on the vector bundle $\pi^*E \cong V \to E$ and $\xi \in T_eE$, show that $\overline{\nabla}_\xi \mathscr{s} = \mathcal{I}_e\mathcal{K}_e\xi$.

(c) Let $\xi \in T_eE$ and $e' \in E_{\pi(e)}$. Show that $\theta_e(\xi)\left(\mathcal{I}_e e'\right) = \left\langle \mathcal{K}_e\xi, e'\right\rangle_{\pi(e)}$.

(d) Show that $d^{\overline{\nabla}}\theta_e(\xi_1,\xi_2)(u) = (\pi^*\mathbf{R})_e(\xi_1,\xi_2)(u,\mathscr{s}(e))$ for $\xi_1,\xi_2 \in T_eE$ and $u \in V_e$.

(e) Let $E_1 \subseteq E$ denote the subbundle of E whose fiber at $p \in M$ consists of the vectors of unit length in E_p. We call it the *sphere bundle* of E. (It is a fiber bundle over M but, naturally, not a vector bundle.) If ξ is tangent to E_1 at $e \in E_1$, show that $\theta_e(\xi)(\mathscr{s}(e)) = 0$.

(f) Consider the $(2m-1,2m)$-form $\Pi_j := \eta \wedge \theta^{2j-1} \wedge \pi^*\mathbf{R}^{m-j}$. Let $\xi_1,\ldots,\xi_{2m-1}$ be tangent vectors to $(E_1)_p$ at $e \in E_1$ (thus the ξ_i are vertical vectors; that is, $\xi_i \in V_e$) and let $X_0, X_1, \ldots, X_{2m-1}$ form a positive orthonormal basis of $(\pi^*E)_e$ such that $X_0 = \mathscr{s}(e)$. Show that

$$\Pi_j(\xi_1,\ldots,\xi_{2m-1})(X_0,\ldots,X_{2m-1}) = 0$$

for $j < m$ and

$$\Pi_m(\xi_1,\ldots,\xi_{2m-1})(X_0,\ldots,X_{2m-1}) = (2m-1)!\,\omega_e(\xi_1,\ldots,\xi_{2m-1})$$

where ω is the volume form on the fiber of E_1 at $p \in M$.

Solution.

(a) For each $e \in E$, both $(\pi^* E)_e$ and V_e are naturally isomorphic to $E_{\pi(e)}$ under the maps $\mathrm{pr}_2 : (\pi^* E)_e \to E_{\pi(e)}$, $\mathrm{pr}_2(e, e') = e'$, and $\mathcal{K}_e : V_e \to E_{\pi(e)}$. Define $\mathcal{I} : \pi^* E \to V$ by

$$(e, e') \in (\pi^* E)_e \mapsto \mathcal{I}(e, e') = \mathcal{I}_e e' \in V_e.$$

It can be checked that $\mathcal{I}$ is a bundle map, a linear isomorphism on each fiber, and smooth. It has an inverse bundle map given by $\mathcal{K} : V \to \pi^* E$ defined for each $\xi \in V_e$ by $\mathcal{K}\xi = (e, \mathcal{K}_e \xi)$, also smooth.

(b) Let $\xi = e'(0) \in T_e E$, where $e(t)$ is a differentiable curve in E such that $e(0) = e$ and $e'(0) = \xi$. Let $\gamma(t) = \pi(e(t))$. Then $e(t)$ is a section of E along $\gamma(t)$ and $\mathcal{s}(t) = (e(t), e(t))$ is a section of $\pi^* E$ along $e(t)$. We have

$$\left.\frac{\overline{\nabla}\mathcal{s}}{dt}\right|_{t=0} = \left(e(0), \left.\frac{\nabla e}{dt}\right|_{t=0} \right) = (e, \mathcal{K}_e \xi).$$

The right-most term is identified with $\mathcal{I}_e \mathcal{K}_e \xi$ by the first part of this exercise.

(c) Let $X \in \mathfrak{X}(E)$ and $U \in \Gamma(\pi^* E) = \Gamma(V)$. Then

$$\theta_e(X)(U) = \left(d^{\overline{\nabla}} \eta \right)(X)(U) = \left(\overline{\nabla}_X \eta \right)_e (U) = X\eta(U) - \eta_e(\overline{\nabla}_X U) = X\langle \mathcal{s}, U \rangle - \langle \mathcal{s}, \overline{\nabla}_X U \rangle_e = \langle \overline{\nabla}_X \mathcal{s}, U \rangle_e = \langle \mathcal{I}_e \mathcal{K}_e X, U \rangle_e.$$

If $U = \mathcal{I}_e e'$ and $X_e = \xi$, then $\theta_e(\xi)(\mathcal{I}_e e') = \langle \mathcal{K}_e \xi, e' \rangle_{\pi(e)}$.

(d) Let $X, Y \in \mathfrak{X}(E)$, $U \in \Gamma(V)$. Then

$$\begin{aligned}
\left(d^{\overline{\nabla}} \theta \right)(X, Y)(U) &= \left(\overline{\nabla}_X \theta(Y) - \overline{\nabla}_Y \theta(X) - \theta([X, Y]) \right)(U) \\
&= X\theta(Y)(U) - \theta(Y)\left(\overline{\nabla}_X U \right) - Y\theta(X)(U) + \theta(X)\left(\overline{\nabla}_Y U \right) - \theta([X, Y])(U) \\
&= X \left\langle \overline{\nabla}_Y \mathcal{s}, U \right\rangle - \left\langle \overline{\nabla}_Y \mathcal{s}, \overline{\nabla}_X U \right\rangle - Y \left\langle \overline{\nabla}_X \mathcal{s}, U \right\rangle + \left\langle \overline{\nabla}_X \mathcal{s}, \overline{\nabla}_Y U \right\rangle - \left\langle \overline{\nabla}_{[X,Y]} \mathcal{s}, U \right\rangle \\
&= \left\langle \overline{\nabla}_X \overline{\nabla}_Y \mathcal{s} - \overline{\nabla}_Y \overline{\nabla}_X \mathcal{s} - \overline{\nabla}_{[X,Y]} \mathcal{s}, U \right\rangle \\
&= \left\langle \overline{R}(X, Y)\mathcal{s}, U \right\rangle
\end{aligned}$$

Writing $\xi_1 = X_e, \xi_2 = Y_e$, $u = U_e$. Then

$$\left(d^{\overline{\nabla}} \theta \right)_e (\xi_1, \xi_2)(u) = \left\langle \overline{R}(\xi_1, \xi_2)\mathcal{s}, u \right\rangle = \left(\pi^* \mathbf{R} \right)(\xi_1, \xi_2)(u, \mathcal{s}(e)).$$

(e) Let $\xi = e'(0) \in T_e E_1$ where $e(t)$ is a smooth curve in E_1 representing ξ. Then $\langle \mathcal{s}(e(t)), \mathcal{s}(e(t)) \rangle = 1$ and

$$0 = \frac{1}{2} \left.\frac{d}{dt}\right|_{t=0} \langle e(t), e(t) \rangle = \left\langle \left.\frac{\nabla e}{dt}\right|_{t=0}, e \right\rangle = \langle \mathcal{K}_e \xi, e \rangle_{\pi(e)} = \theta_e(\xi)(\mathcal{s}(e)).$$

(f) If $j < m$, the form Π_j contains a factor $\pi^* \mathbf{R}$. But this term vanishes when evaluated on a vertical vector ξ_j. Thus $\Pi_j(\xi_1, \ldots, \xi_{2m-1})(X_0, \ldots, X_{2m-1}) = 0$ if $j < m$. Let us now look at $\Pi_m = \eta \wedge \theta \wedge \cdots \wedge \theta = \eta \wedge \theta^{2m-1}$. Keep in mind that $\theta(\xi)(X_0) = \theta(\xi)(\mathscr{s}) = 0$ and $\eta_e(\mathscr{s}(e)) = \langle e, e\rangle = 1$ for $e \in E_1$.

$$\begin{aligned}\Pi_m(\xi_1,\dots,\xi_{2m-1})(X_0,\dots,X_{2m-1}) &= \sum_{\sigma\in S_{2m-1}} \operatorname{sign}(\sigma)\left(\eta\wedge\theta\left(\xi_{\sigma(1)}\right)\wedge\cdots\wedge\theta\left(\xi_{\sigma(2m-1)}\right)\right)(X_0,\dots,X_{2m-})\\ &= (2m-1)!\left(\eta\wedge\theta\left(\xi_1\right)\wedge\cdots\wedge\theta\left(\xi_{2m-1}\right)\right)(X_0,\dots,X_{2m-1})\\ &= (2m-1)!\sum_{\sigma\in S_{2m}} \operatorname{sign}(\sigma)\eta\left(X_{\sigma(0)}\right)\left[\theta(\xi_1)\left(X_{\sigma(1)}\right)\right]\cdots\left[\theta(\xi_{2m-1})\left(X_{\sigma(2m-1)}\right]\right)\\ &= (2m-1)!\sum_{\sigma\in S_{2m}:\sigma(0)=0} \operatorname{sign}(\sigma)\left[\theta(\xi_1)\left(X_{\sigma(1)}\right)\right]\cdots\left[\theta(\xi_{2m-1})\left(X_{\sigma(2m-1)}\right]\right).\end{aligned}$$

The vectors ξ are tangent to the fiber $(E_1)_p$ at $e \in E_1$ and the X_i lie in $\pi * E \cong V$. So we may write $X_i = \mathscr{I}_e e_i$ where the e_i are tangent to E at p and form a positive orthonormal basis. Since the X_i are orthogonal to $X_0 = \mathscr{s}$ for $i \geq 1$, the e_i are orthogonal to e for $i \geq 1$. This means that the $e_1,\dots,e_{2m-1}$ constitute a positive orthonormal basis for the tangent space to $(E_1)_p$ at e. Observe that $\theta(\xi_i)(X_j) = \langle \mathscr{I}\mathscr{K}\xi_i, \mathscr{I}e_j\rangle_e = \langle \xi_i, \mathscr{I}e_j\rangle_e$. Consequently, the above alternating sum gives $\omega_e(\xi_1,\dots,\xi_{2m-1})$ and we conclude that

$$\Pi_m(\xi_1,\dots,\xi_{2m-1})(X_0,\dots,X_{2m-1}) = (2m-1)!\omega_e(\xi_1,\dots,\xi_{2m-1}).$$

⋄

7. **Pull-back of the Chern-Euler form to E_1.** Keeping with the notation of the previous exercise, define

$$\Pi := \sum_{j=1}^m c_j\Pi_j,$$

where

$$c_j = -\frac{(j-1)!}{2^{m-j+1}(m-j)!(2j-1)!\pi^m}.$$

Then $\Pi \in \Gamma\left(\Lambda^{2m-1}(T^*E)\otimes\Lambda^{2m}(\pi^*E^*)\right)$. Show that $d^{\overline{\nabla}}\Pi = \pi^*\mathbf{e}$ on E_1.

Note that Π and $\pi^*\mathbf{e}$ are forms on E (or E_1) with coefficients in the vector bundle $\Lambda^{2m}(E^*)$. Since the latter bundle is trivial for an oriented E, with the volume form ω serving as a global section, we can write Π and $\pi^*\mathbf{e}$ as a product of ordinary forms on the base manifold E_1 tensor ω. Using the same notations Π and $\pi^*\mathbf{e}$ for these ordinary forms, then $\pi^*\mathbf{e} = d\Pi$, where d is now the ordinary exterior derivative. Another point to observe is that the integral of Π over $(E_1)_p$ is -1. This is due to the previous item and the choice of the constant c_m.

Solution. This is a tedious calculation which you can find in Section 3.67, pages 142 and 143 of Walter Poor's text. Here are the main steps. For simplicity we write ∇ for $\overline{\nabla}$. Since $d^\nabla\mathbf{R} = 0$ and $d^\nabla\eta = \theta$, we have

$$\begin{aligned}d^\nabla\Pi_j &= d^\nabla\left(\eta\wedge\theta^{2j-1}\wedge\pi^*\mathbf{R}^{m-j}\right)\\ &= d^\nabla\eta\wedge\theta^{2j-1}\wedge\pi^*\mathbf{R}^{m-j} + \eta\wedge d^\nabla\left(\theta^{2j-1}\right)\wedge\pi^*\mathbf{R}^{m-j} + \eta\theta^{2j-1}\wedge d^\nabla\pi^*\mathbf{R}^{m-j}\\ &= \theta^{2j}\wedge\pi^*\mathbf{R}^{m-j} + (2j-1)\theta^{2j-2}\wedge\eta\wedge d^\nabla\theta\wedge\pi^*\mathbf{R}^{m-j}.\end{aligned}$$

For the last line, keep in mind that η is a $(0,1)$-form, θ is a $(1,1)$-form, and $d^\nabla\theta$ is a $(2,1)$-form and use previous formulas for wedge product and exterior derivative. Going back to the relationship between $d^\nabla\theta$ and $\pi^*\mathbf{R}$, it is possible to show (see W. Poor's text, page 143) that

$$\theta^{2j-2} \wedge \eta \wedge d^{\nabla}\theta \wedge \pi^* \mathbf{R}^{m-j} = -\frac{1}{m-j+1}\theta^{2j-2} \wedge \pi^* \mathbf{R}^{m-j+1}.$$

Therefore

$$d^{\nabla}\Pi_j = \theta^{2j} \wedge \pi^* \mathbf{R}^{m-j} - \frac{2j-1}{m-j+1}\theta^{2j-2} \wedge \pi^* \mathbf{R}^{m-j+1}.$$

We thus obtain

$$\begin{aligned} d^{\nabla}\Pi &= \sum_{j=1}^{m} c_j d^{\nabla}\Pi_j \\ &= \sum_{j=1}^{m} c_j \left(\theta^{2j} \wedge \pi^* \mathbf{R}^{m-j} - \frac{2j-1}{m-j+1}\theta^{2j-2} \wedge \pi^* \mathbf{R}^{m-j+1}\right) \\ &= -\frac{c_1}{m}\pi^* \mathbf{R}^m + \sum_{j=1}^{m-1} (-1)^{m-j} \left(c_j + \frac{2j+1}{m-j} c_{j+1}\right)\theta^{2j} \wedge \pi^* \mathbf{R}^{m-j} + c_m \theta^{2m} \\ &= \pi^* \left(\frac{\mathbf{R}^m}{2^m \pi^m m!}\right) + c_m \theta^{2m} \\ &= \pi^* \mathbf{e}. \end{aligned}$$

At the last step we used that $\theta^{2m} = 0$ on E_1.

8. **End of the proof of the Gauss-Bonnet-Chern theorem.** At this point, let $E = TM$ and E_1 the sphere bundle T^1M. The rest of the proof of the Gauss-Bonnet-Chern theorem follows a similar pattern used in dimension 2, with the Poincaré-Hopf theorem playing a central role.

   Let $\mathbf{e}$ be the Chern-Euler form on a compact, oriented $2m$-dimensional Riemannian manifold M and let

   $$\Pi \in \Omega^{2m-1}\left(T^1M, \Lambda^{2m}\pi^* T^* M\right)$$

   be the above defined form, so that $d^{\nabla}\Pi = \pi^* \mathbf{e}$ on S_M.

   Fix a vector field X on M with finitely many zeros, and let $Z \subset M$ be the zero set of X. Let $Y = X/\|X\|$, a smooth section of the sphere bundle T^1M' on the complement M' of Z. We may choose X so that the limit $Y \circ \gamma(t)$ as $t \to 0+$ exists for every geodesic γ issuing from each zero of X.

   There exists $\epsilon > 0$ such that for each zero $p \in Z$ of X, the set $\{v \in T_pM : \|v\| \le \epsilon\}$ is mapped diffeomorphically by $\exp_p$ to the closure $\overline{B_\epsilon(p)}$ in M of the geodesic ball $B_\epsilon(p)$ about p of radius ϵ, and such that the distance in M between any two zeros of X is bigger than 2ϵ. Fix $p \in Z$ and let F be the unit sphere T^1_pM in T_pM. Define $\varphi : F \times [0,\epsilon] \to M$ by $\varphi(v,t) := \exp_p(tv)$. The image of φ is $\overline{B_\epsilon(p)}$ and φ maps $F \times (0,\epsilon]$ diffeomorphically to $\overline{B_\epsilon(p)} \setminus \{p\}$.

   The smooth map $Y \circ \varphi : F \times (0,\epsilon] \to T^1M$ is a section of the sphere bundle along φ. It has a unique extension to a section $W : F \times [0,\epsilon] \to T^1M$ along φ, even though Y cannot be extended continuously to $B_\epsilon(p)$. The extension W maps $F \cong F \times \{0\}$ differentiably to $T^1_pM = F$, and since p is the only zero of X in $\overline{B_\epsilon(p)}$, the degree of the map W from F to F equals the degree of the map $d(\exp_p^{-1}) \circ Y \circ \exp_p$ from F to F. This degree is called the *index of the vector field* X at $p \in M$.

   The pull-back $\pi^*\mathbf{e}$ to T^1M of the Chern-Euler form on M equals the exterior covariant derivative $d^{\nabla}\Pi$ of the form Π defined above in the previous problem. Using the volume form on the bundle $(\pi \circ \mathrm{inc})^* TM$ over T^1M, we can identify $\pi^*\mathbf{e}$ and Π with ordinary differential forms on T^1M and can write $d\Pi = \pi^*\mathbf{e}$. Since Y is a section of T^1M', on M' the Chern-Euler form $\mathbf{e}$ equals

$$\mathbf{e} = (\pi\circ Y)^*\mathbf{e} = Y^*\pi^*\mathbf{e} = Y^*d\Pi = dY^*\Pi.$$

Hence

$$\begin{aligned}\int_{\overline{B_\epsilon(p)}}\mathbf{e} &= \int_{\overline{B_\epsilon(p)}\setminus\{p\}}\mathbf{e} = \int_{\varphi(F\times(0,\epsilon])} dY^*\Pi = \int_{F\times(0,\epsilon]} d(Y\circ\varphi)^*\Pi = \int_{F\times[0,\epsilon]} dW^*\Pi \\ &= \int_{\partial(F\times[0,\epsilon])} W^*\Pi = \int_{F\times\{\epsilon\}} W^*\Pi - \int_{F\times\{0\}} W^*\Pi \\ &= \int_{\partial B_\epsilon(p)} Y^*\Pi - \int_F W^*\Pi \\ &= \int_{\partial B_\epsilon(p)} Y^*\Pi - \deg(W)\int_F \Pi.\end{aligned}$$

Recall from HW 9 that the volume of the sphere S^{2m-1} is $2\pi^m/(m-1)!$. Therefore

$$\int_F \Pi = \int_F c_m\Pi_m = c_m(2m-1)!\int_F \omega_e = -\frac{(m-1)!}{2(2m-1)!\pi^m}(2m-1)!\mathrm{Vol}(S^{2m-1}) = -1$$

hence

$$\int_{\overline{B_\epsilon(p)}}\mathbf{e} = \int_{\partial B_\epsilon(p)} Y^*\Pi + \mathrm{Ind}(X,p).$$

Set $N := \bigcup_{p\in Z} B_\epsilon(p)$. Then

$$\begin{aligned}\int_M \mathbf{e} &= \int_{M\setminus N}\mathbf{e} + \sum_{p\in Z}\int_{\overline{B_\epsilon(p)}}\mathbf{e} \\ &= \int_{\partial(M\setminus N)} Y^*\Pi + \sum_{p\in Z}\left(\int_{\partial B_\epsilon(p)} Y^*\Pi + \mathrm{Ind}(X,p)\right) \\ &= -\int_{\partial N} Y^*\Pi + \int_{\partial N} Y^*\Pi + \sum_{p\in Z}\mathrm{Ind}(X,p) \\ &= \mathrm{Index}(X).\end{aligned}$$

By the Poincaré-Hopf theorem, the index of X equals the Euler characteristic of M.

9. **The Hopf bundle and Chern number.** In what follows, elements of $\mathbb{C}P^1$ will be written as $[z] = [z_1 : z_2]$ (so-called *homogeneous coordinates*) to indicate the equivalence class represented by $z \in \mathbb{C}^2\setminus\{(0,0)\}$ under the action of the multiplicative group of non-zero complex numbers. (Or, equivalently, $z \in S^3$ under the action of $U(1)$.)

   The Hopf bundle is the bundle of unit length vectors of a complex line bundle over $\mathbb{C}P^1$ called the *tautological bundle*, which is a rank 1 (complex) subbundle L of the trivial bundle $\mathrm{pr}_1 : \mathbb{C}P^1\times\mathbb{C}^2 \to \mathbb{C}P^1$ defined by the property that the fiber $L_{[z]}$ of L at $[z]$ the one-dimensional complex subspace in $\mathbb{C}^2$ spanned by z itself (thus the name *tautological*). I'll used the same letter π to denote the base point map $\pi : L \to \mathbb{C}P^1$. By definition,

$$L = \{([z],\lambda z) : [z]\in\mathbb{C}P^1, \lambda\in\mathbb{C}\} = \{(\ell,z)\in\mathbb{C}P^1\times\mathbb{C}^2 : z\in\ell\},\ \ \pi(\ell,z) = \ell.$$

   The trivial vector bundle $\mathbb{C}P^1\times\mathbb{C}^2$ can be endowed with a Hermitian metric by giving $\mathbb{C}^2$ the standard (complex valued) inner product

$$\langle(z_1,z_2),(w_1,w_2)\rangle = \bar{z}_1 w_1 + \bar{z}_2 w_2.$$

(Note: as a matter of taste, I deviate from the textbook convention by placing the complex conjugate bar over the first vector argument of the inner product.) The Hopf bundle is then the bundle of unit length vectors of L. (It is a *principal bundle* over $\mathbb{C}P^1$ with structure group $U(1)$, according to the definitions at the beginning of Chapter 6 of the textbook, although we don't need this fact for this assignment.) We thus have $S^3 \subseteq L \subseteq \mathbb{C}P^1 \times \mathbb{C}^2$, and these are all bundles over projective space.

Our goal is to compute the *Chern number* of L, which is the integral over $S^2 \cong \mathbb{C}P^1$ of the Chern class $c_1(L)$.

The line bundle L with the Hermitian metric $\langle\cdot,\cdot\rangle$ can be given a metric connection as follows. For each $[z] \in \mathbb{C}P^1$, we have the orthogonal direct sum decomposition $\mathbb{C}^2 = L_{[z]} \oplus L_{[z]}^{\perp}$ where $L_{[z]}$ is the fiber of L over $[z]$ and

$$L_{[z]}^{\perp} = \left\{u \in \mathbb{C}^2 : \langle z, u\rangle = 0\right\}.$$

Let $\Pi_{[z]} : \mathbb{C}^2 \to L_{[z]}$ be the resulting projection map. We now define a connection ∇ on smooth sections of L as follows: given $\xi \in \Gamma(L)$ and $u \in T_{[z]}\mathbb{C}P^1$,

$$\nabla_u \xi = \Pi_{[z]} D_u \xi \tag{5}$$

where D is the ordinary derivative of a $\mathbb{C}^2$-valued (equivalently, $\mathbb{R}^4$-valued) function on $\mathbb{C}P^1$:

$$D_u(f_1 + i f_2, f_3 + i f_4) = \left(u f_1 + i u f_2, u f_3 + u f_4\right).$$

We will need the connection and curvature forms of ∇ for convenient choices of local trivializations of L. But first note that if $\xi : U \to S^3 \subseteq L$ is a section of L over $U \subseteq \mathbb{C}P^1$, then the corresponding connection form is $\omega_p(u) = \langle \xi(p), \nabla_u \xi\rangle = \langle \xi(p), D_u\xi\rangle$ for $u \in T_p\mathbb{C}P^1$. This being an actual 1-form on U, the associated curvature form is simply $\Omega = d\omega + \omega\wedge\omega = d\omega$. The effect of taking sections of L of unit length (hence in S^3) is that $\mathrm{Re}\left(\langle \xi, D_u\xi\rangle\right) = 0$. In fact,

$$0 = u\langle \xi,\xi\rangle = \langle \nabla_u\xi, \xi\rangle + \langle \xi, \nabla_u \xi\rangle = \langle \xi, \nabla_u\xi\rangle + \overline{\langle \xi, \nabla_u\xi\rangle} = 2\mathrm{Re}\left(\langle \xi, \nabla_u\xi\rangle\right).$$

Therefore

$$\omega_p(u) = 2i\,\mathrm{Im}\left(\langle \xi(p), D_u\xi\rangle\right). \tag{6}$$

Observe that if η is another section also defined on U, then $\eta = f\xi$ where f is a complex-valued function on U of unit modulus: $|f(p)| = 1$. Then $\omega_\eta = \omega_\xi + f^{-1}df$ and $d\omega_\eta = d\omega_\xi$. This means that the curvature form Ω will define a global closed form on $\mathbb{C}P^1$. The Chern class is then (according to the definition on page 235 of the textbook in the special case of rank 1) $c_1(L) = \frac{i}{2\pi}\Omega$ and the Chern number, our ultimate goal here, is the value of $\int_{\mathbb{C}P^1} c_1(L)$.

We now choose trivializing neighborhoods and sections of the Hopf bundle. Let $\{U_-, U_+\}$ be the open cover of $\mathbb{C}P^1$ given by

$$U_- := \left\{[z_1 : z_2] \in \mathbb{C}P^1 : z_2 \neq 0\right\},\ \ U_+ := \left\{[z_1 : z_2] \in \mathbb{C}P^1 : z_1 \neq 0\right\}.$$

We define coordinates $\varphi_\pm : U_\pm \to \mathbb{C}$ by $\varphi_-([z_1 : z_2]) = z_- := z_1/z_2$ and $\varphi_+([z_1 : z_2]) = z_+ := z_2/z_1$. (These give us a smooth (in fact, holomorphic) atlas on $\mathbb{C}P^1$.) On $U_\pm$ we define the section $\xi_\pm$ as follows:

$$\xi_-([z_1 : z_2]) := \frac{1}{\sqrt{1+|z_-|^2}}\begin{pmatrix} z_- \\ 1\end{pmatrix},\ \ \xi_+([z_1 : z_2]) := \frac{1}{\sqrt{1+|z_+|^2}}\begin{pmatrix} 1 \\ z_+\end{pmatrix} \tag{7}$$

Associated to these sections we have connection forms ω_-, ω_+. You will show in one of the exercises below that on $U_- \cap U_+$

$$\omega_+ = \omega_+ + \overline{f}\,df \tag{8}$$

where $f([z_1:z_2]) = z_+/|z_+|$.

◇

Before moving forward, let us try to get a better sense of what parts of S^2 are covered by $U_\pm$. Let us define the map $F:\mathbb{C}P^1 \to S^2$ such that $F([z_1:z_2]) = (x_1,x_2,x_3)$ where

$$x_1 = 2\mathrm{Re}\left(z_1\overline{z}_2\right), \;\; x_2 = 2\mathrm{Im}\left(z_1\overline{z}_2\right), \;\; x_3 = |z_1|^2 - |z_2|^2.$$

We are assuming here that $|z_1|^2+|z_2|^2 = 1$. It is not difficult to see from this expression that F is injective. To see that it is surjective, observe that

$$F\left(\left[\cos\left(\frac{\varphi}{2}\right)e^{\frac{\psi+\theta}{2}i} : \sin\left(\frac{\varphi}{2}\right)e^{\frac{\psi-\theta}{2}i}\right]\right) = \left(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\right).$$

On the right-hand side we have an arbitrary point on S^2 expressed in spherical coordinates by allowing $\theta \in [0,2\pi)$ and $\varphi \in [0,\pi]$. Changing ψ does not change the point $[z_1:z_2]$ (ψ parametrizes the $U(1)$ fiber of the Hopf bundle). From this description we see that $[1:0]$ is mapped to the North Pole $N = (0,0,1)^\intercal$ and $[0:1]$ to the South Pole $S = (0,0,-1)^\intercal$, while $\left[\frac{1}{\sqrt{2}} : \frac{e^{i\theta}}{\sqrt{2}}\right]$ is mapped to the equator ($x_3 = 0$). Also note that the function f in (8) maps $\left[\frac{1}{\sqrt{2}} : \frac{e^{i\theta}}{\sqrt{2}}\right]$ to $e^{i\theta}$. We have that (U_+,φ_+) parametrizes a region of S^2 that contains the northern hemisphere S^2_+ while (U_-,φ_-) parametrizes a region containing the southern hemisphere S^2_-. As θ increases in $\left[\frac{1}{\sqrt{2}} : \frac{e^{i\theta}}{\sqrt{2}}\right]$ we traverse the equation in the positive direction if we regard the equation as the boundary of the northern hemisphere, and in the negative direction relative to the boundary orientation of the southern hemisphere. Also observe that $f\left(\left[\frac{1}{\sqrt{2}} : \frac{e^{i\theta}}{\sqrt{2}}\right]\right) = e^{i\theta}$ and that the pull-back (or restriction) of $\overline{f}df$ to the equator is

$$e^{-i\theta}d\left(e^{i\theta}\right) = e^{-i\theta}e^{i\theta}id\theta = id\theta.$$

Finally observe that

$$\int_{\mathbb{C}P^1}\Omega = \int_{S^2_+}d\omega_+ + \int_{S^2_-}d\omega_- = \int_{\partial S^2_+}\omega_+ + \int_{\partial S^2_-}\omega_- = \int_{\partial S^+}(\omega_+ - \omega_-) = \int_0^{2\pi} id\theta = i2\pi.$$

We conclude that

$$\int_{\mathbb{C}P^1} c_1(L) = \int_{\mathbb{C}P^1}\frac{i\Omega}{2\pi} = \frac{i}{2\pi}2\pi i = -1.$$

Therefore the Chern number of the tautological line bundle over $\mathbb{C}P^1$ is -1.

◇

(a) Let $L^{\otimes n}$ be the line bundle over $\mathbb{C}P^1$ given by the tensor product $L\otimes\cdots\otimes L$ (n times). Show that the Chern number of $L^{\otimes n}$ is $-n$. (As a lemma, show that if L_1, L_2 are two line bundles then the first Chern class of their tensor product is $c_1(L_1\otimes L_2) = c_1(L_1) + c_1(L_2)$.)

(b) Let L^* denote the dual bundle to the tautological line bundle L. Show that $c_1(L^*) = -c_1(L)$. In particular, L^* has Chern number 1. (Note: $L\otimes L^*$ is the bundle of endomorphisms of L. This line bundle has a nowhere vanishing section given by the identity map on each fiber of L.)

Knowing that the trivial line bundle has Chern number 0, we conclude from the above that every integer $n\in\mathbb{Z}$ is the Chern number of some complex line bundle over $\mathbb{C}P^1$ (and line bundles with different Chern numbers are not isomorphic).

Solution.

(a) If ξ_1 and ξ_2 are local sections of the line bundles L_1 and L_2, and ω_1 and ω_2 are the respective connection forms, then $\xi_1 \otimes \xi_2$ is a local section of $L_1 \otimes L_2$ whose connection form can be found as follows:

$$\nabla_u(\xi_1 \otimes \xi_2) = (\nabla_u \xi_1) \otimes \xi_2 + \xi_1 \otimes (\nabla_u \xi_2) = \omega_1(u)\xi_1 \otimes \xi_2 + \omega_2(u)\xi_1 \otimes \xi_2 = (\omega_1(u) + \omega_2(u))\xi_1 \otimes \xi_2.$$

Thus the product bundle has connection form $\omega_1 + \omega_2$. In particular, the curvature forms, which are the globally defined closed 2-forms $c_1(L_i) = d\omega_i$, satisfy

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2).$$

By a simple induction we obtain that $c_1(L^{\otimes n}) = nc_1(L)$. For the tautological line bundle over $\mathbb{C}P^1$, we proved above that $\int_{\mathbb{C}P^1} c_1(L) = -1$, so $\int_{\mathbb{C}P^1} c_1(L^{\otimes n}) = -n$ as claimed.

(b) Using the result of the previous item (concerning the Chern class of a tensor product of line bundles) and the observation that $L \otimes L^*$ is trivial, we obtain: $0 = c_1(L^*) + c_1(L)$ as claimed.

◇