Homework set 9 - due 11/15/20

Math 5047 – Renato Feres

Turn in the solutions to problems 6, 7, 8(a,b,c,d), and 9(b).

1. **Reading: The Gauss-Bonnet theorem.** We consider here an oriented 2-dimensional Riemannian manifold M. Orientation is not essential—one can do most of the work on the orientation double cover of M.

   Definition 1 (Geodesic curvature)**.** *Let $c : (a, b) \to M$ be a smooth curve parametrized by arclength. Denote by $N(s)$ the unit vector perpendicular to $T(s) = c'(s)$ such that $T(s), N(s)$ form a positive orthonormal basis of $T_{c(s)}M$ for each s. Then the number*
   $$\kappa_g(s) := \left\langle \frac{\nabla c'}{ds}, N(s) \right\rangle$$
   is called the geodesic curvature *of c.*

   Note that $c(s)$ is a geodesic if and only if its geodesic curvature is zero.

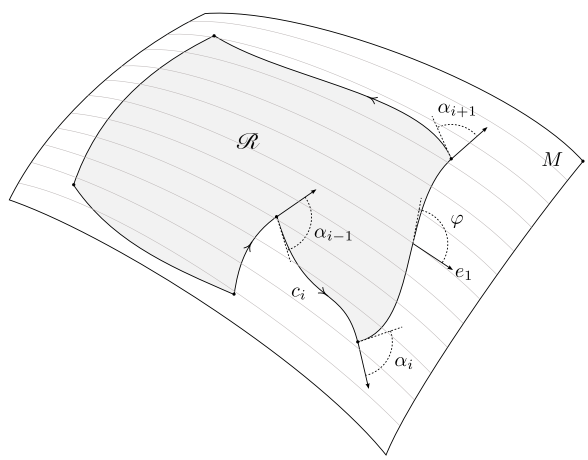

Figure 1: Definitions for the local version of the Gauss-Bonnet theorem.

Consider a region $\mathscr{R}$ homeomorphic to a disc in the oriented surface M. We suppose that the boundary of $\mathscr{R}$ is a continuous, piecewise smooth simple closed curve c with finitely many smooth curve segments as indicated in Figure **??**. The smooth segments of c are denoted $c_i : [a_i, b_i] \to M$ and are parametrized by arclength. Also suppose that $\mathscr{R}$ is contained in an open set $U \subset M$ on which is defined a smooth vector field e_1 of unit length. Let e_2 be the unit length vector field orthogonal to e_1 such that e_1, e_2 form at each $p \in U$ a positive basis of T_pM. At each point $c_i(s)$ let $\varphi(s)$ be the angle that $c_i'(s)$ makes with $e_1(c_i(s))$ as indicated in Figure **??**. We call $\varphi(s)$ the *turning angle*. At the corner point connecting c_i and c_{i+1}, where c is not differentiable, let α_i be the angle between $c_i'(b_i)$ and $c_{i+1}'(a_{i+1})$ measured counterclockwise. Note that the accumulated turning angle that

c' makes with e_1 as the closed curve is traced around once, including the jump discontinuities at the corners of $\mathcal{R}$, must be 2π. I will take this as a fact and omit the proof. In other words, if $\varphi_i(s)$ is the turning angle along $c_i(s)$, then

$$\sum_i \left[\int_{a_i}^{b_i} \varphi_i'(s)\,ds + \alpha_i \right] = 2\pi. \tag{1}$$

Theorem 2 (Gauss-Bonnet, local version)**.** *Let $(M, \langle\cdot,\cdot\rangle)$ be a 2-dimensional, oriented Riemannian manifold and let $\mathcal{R} \subset M$ be a compact region as indicated in Figure* **??**. *The boundary of $\mathcal{R}$ is a piecewise smooth curve with smooth segments $c_i : [a_i, b_i] \to M$ with turning angle $\varphi_i(s)$ having jump discontinuities at the corners given by α_i as indicated in the figure. Then*

$$\iint_{\mathcal{R}} K\,dA + \int_c \kappa_g(s)\,ds + \sum_i \alpha_i = 2\pi$$

where K is the Gauss curvature and k_g is the geodesic curvature.

Proof. Let us recall a few facts. The Gauss curvature $K(p)$ is given, for any orthonormal basis u, v of T_pM, by

$$K(p) = \langle R_p(u,v)v, u \rangle.$$

Cartan's structure equations expressed with respect to a local orthonormal frame e_1, e_2 and its dual frame θ_1, θ_2 give

$$\langle R_p(u,v)v, u \rangle = \Omega_{12}(e_1, e_2) = d\omega_{12}(e_1, e_2)$$

where $\omega_{ij}(u) = \langle e_i, \nabla_u e_j \rangle$. Thus

$$d\omega_{12} = K(p)\,\theta_1 \wedge \theta_2$$

where $\theta_1 \wedge \theta_2$ is the area form of M corresponding to the area measure dA. At points where $c'(s)$ exists we have (dropping for now the index i to simplify the notation)

$$c'(s) = \cos(\varphi(s))e_1(c(s)) + \sin(\varphi(s))e_2(c(s)).$$

The unit normal vector field $N(s)$ obtained by rotating $c'(s)$ counterclockwise by $\pi/2$ is

$$N(s) = -\sin(\varphi(s))e_1(c(s)) + \cos(\varphi(s))e_2(c(s)).$$

Keeping in mind that $\langle \nabla e_i/ds, e_i \rangle = 0$, we have

$$\frac{\nabla c'}{ds} = \left(-\varphi' \sin(\varphi) + \sin(\varphi)\left\langle \frac{\nabla e_2}{ds}, e_1 \right\rangle\right) e_1 + \left(\varphi' \cos(\varphi) + \cos(\varphi)\left\langle \frac{\nabla e_1}{ds}, e_2 \right\rangle\right) e_2.$$

The normal component of this vector is then

$$\kappa_g = \left\langle \frac{\nabla c'}{ds}, N \right\rangle = \varphi' + \left\langle \frac{\nabla e_1}{ds}, e_2 \right\rangle = \varphi' - \omega_{12}(c').$$

Integrating over each c_i and summing over i (reintroducing the index at this point) gives

$$\sum_i \left(\int_{a_i}^{b_i} \kappa_g(s)\,ds + \int_{c_i} \omega_{12} \right) = \sum_i \int_{a_i}^{b_i} \varphi'(s)\,ds = 2\pi - \sum_i \alpha_i.$$

We now apply Stokes' theorem:

$$\sum_i \int_{c_i} \omega_{12} = \iint_{\mathcal{R}} d\omega_{12} = \iint_{\mathcal{R}} K\,dA.$$

This gives

$$\iint_{\mathcal{R}} K\,dA + \int_c \kappa_g(s)\,ds = 2\pi - \sum_i \alpha_i$$

as we wished to show. □

Corollary 3 (Geodesic n-gon)**.** *Let $\mathcal{R}$ be as in the statement of Theorem* **??** *with the additional assumption that the smooth segments of the boundary of this region are geodesics. Let as before α_i be the turning angles at the corners of $\mathcal{R}$. Then*

$$\iint_{\mathcal{R}} K\,dA = 2\pi - \sum_i \alpha_i.$$

Suppose now that M is a compact, oriented 2-dimensional Riemannian manifold empty boundary.

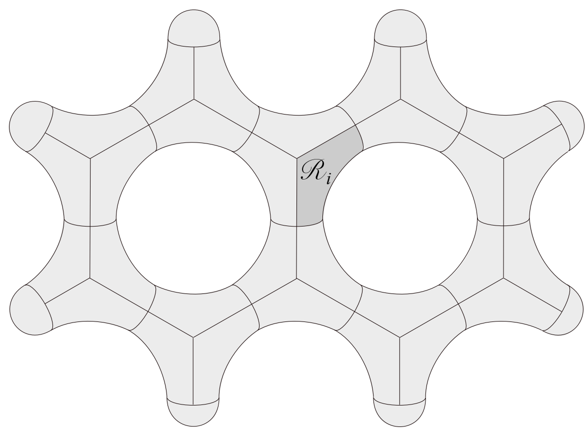

Figure 2: Decomposition of M into regions $\mathcal{R}_i$ for the global Gauss-Bonnet theorem. The edge curves do not need to be geodesics as in this figure. Note that the spherical caps are regions bounded by 4 sides with corner angles equal to π.

We may partition M into finitely many nonoverlapping regions $\mathcal{R}_i$, each of the type considered in the local form of the Gauss-Bonnet theorem. The regions are assumed to intersect only on parts of their boundary. Each boundary segment, or edge, is shared by exactly two regions. Let α_{ij} denote the turning angles at the corners of the boundary of $\mathcal{R}_i$ and β_{ij} the corresponding internal angles, so that $\alpha_{ij} = \pi - \beta_{ij}$. Let $\#_f$ be the number of regions, $\#_e$ the number of edges and $\#_v$ the number of corner vertices of the decomposition.

Definition 4 (Euler characteristic)**.** *The number $\chi(M) = \#_v - \#_e + \#_f$ is known as the* Euler characteristic *of the surface.*

It is known that the Euler characteristic $\chi(M)$ does not depend on the decomposition M into regions $\mathcal{R}_i$.

Theorem 5 (Gauss-Bonnet, global version)**.** *Let M be a compact, two-dimensional, orientable Riemannian manifold without boundary. Then*

$$\frac{1}{2\pi}\iint_M K\,dA = \chi(M)$$

where $\chi(M) \in \mathbb{Z}$ is the Euler characteristic of M. In particular, $\chi(M)$ does not depend on the decomposition of M into regions $\mathcal{R}_i$ and the integral on the left-hand side does not depend on the Riemannian metric from which K is calculated.

Proof. We use the decomposition $M = \bigcup_i \mathcal{R}_i$. Note that $\sum_{i,j} \beta_{ij} = 2\pi \#_v$ and $\sum_i \int_{\partial\mathcal{R}_i} \kappa_g(s)\,ds = 0$. The latter is because each edge is shared by a pair of regions that contribute geodesic curvature integrals with opposite signs. Then, using the local form of the Gauss-Bonnet theorem,

$$\iint_M K\,dA = \sum_i \iint_{\mathcal{R}_i} K\,dA = \sum_i \left(\iint_{\mathcal{R}_i} K\,dA + \int_{\partial \mathcal{R}_i} \kappa_g(s)\,ds \right) = \sum_i \left(2\pi - \sum_j \alpha_{ij} \right) = 2\pi \#_f - \sum_{ij} \alpha_{ij}.$$

The sum $\sum_{ij}$ is over all the edges counted with multiplicity two, so the sum has $2\#_e$ terms. The internal angles β_{ij} add up to $2\pi\#_v$ since each vertex accounts for an angle 2π. Therefore,

$$\sum_{ij} \alpha_{ij} = \sum_{ij} (\pi - \beta_{ij}) = 2\pi\#_e - \sum_{ij} \beta_{ij} = 2\pi\#_e - 2\pi\#_v.$$

Therefore

$$\iint_M K\,dA = 2\pi(\#_f - \#_e + \#_v) = 2\pi\chi(M).$$

as we wished to prove. □

A fundamental theorem in topology states that two compact surfaces without boundary (not necessarily submanifolds of $\mathbb{R}^3$) are homeomorphic to one another if and only if their Euler characteristics coincide and both surfaces are either orientable or non-orientable. Orientable, compact surfaces without boundary are determined up to homeomorphism (equivalently, diffeomorphism) if they have the same Euler characteristics. The model surface with Euler characteristic $\chi(M) = 2 - 2g$ is homeomorphic to a sphere with g handles attached.

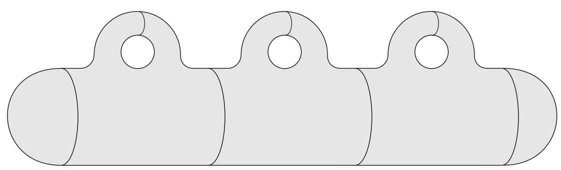

Figure 3: A surface of genus $g = 3$ and Euler characteristic $\chi(M) = 2 - 2g = -4$.

We will later revisit the proof of the Gauss-Bonnet theorem for surfaces in light of Chern's proof of the general Chern-Gauss-Bonnet theorem.

2. **Read Chapter 9, pages 263 to 282, of Lee's text, about the Gauss-Bonnet Theorem for surfaces.**

3. **Curvature of an regular plane curve.** Let $c(t) = (x(t), y(t))$ be a regular plane curve, where t is not necessarily arclength parameter. Starting from the definition of the signed curvature κ given above (Definition **??**), show that

   (a) $\kappa = \left(\dot{x}\ddot{y} - \dot{y}\ddot{x}\right) / \left(\dot{x}^2 + \dot{y}^2\right)^{3/2}$. Here $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$.

   (b) Obtain the curvature $\kappa(\theta)$ of an ellipse given in parametric form by $c(\theta) = (a\cos\theta, b\sin\theta)$.

4. **Shape operator, mean and Gaussian curvatures of surfaces of revolution.** Let $(f(u), g(u))$ be a unit-speed curve (i.e., a curve parametrized by arc length) without self-intersections in the (y,z)-plane. Assume $f(u) > 0$, so that $(f(u), g(u))$ can be rotated about the z-axis to form a surface of revolution M in $\mathbb{R}^3$. A parametrization of the surface of revolution is

$$\Psi(u,v) = \begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ g(u) \end{pmatrix}, \quad 0 < v < 2\pi.$$

Assume that Ψ is a diffeomorphism onto its image, so that u, v are coordinates on a chart U in M. Then

$$e_1 = \partial/\partial u, \quad e_2 = \partial/\partial v$$

form a basis for the tangent space T_pM for $p \in U$.

(a) Find the matrix of the shape operator of the surface of revolution with respect to the basis e_1, e_2 at p.

(b) Compute the mean and Gaussian curvatures of M at p. (Recall: the Gaussian curvature is the product of the principal curvatures and the mean curvature is the arithmetic average of the principal curvatures. The principal curvatures are the eigenvalues of the shape operator.)

5. **Geometry of tubes in $\mathbb{R}^3$.** Let $c(s)$ be a smooth closed curve in $\mathbb{R}^2$ parametrized by arc-length, where $\mathbb{R}^2$ is regarded as a plane in $\mathbb{R}^3$. Let $e_1(s) := c'(s)$ (the tangent vector to the curve) and $e_2(s)$ the unit orthogonal vector to $e_1(s)$ so that $e_3 := e_1(s) \times e_2(s)$ is the standard basis vector $(0,0,1)$ of $\mathbb{R}^3$. Define the parametric surface $M \subseteq \mathbb{R}^3$ with parametrization

$$\Phi(s,\varphi) = c(s) + r\left[\cos(\varphi)e_2(s) + \sin(\varphi)e_3\right]$$

for $0 \le \varphi \le 2\pi$. Note that the image of Φ is a tube of radius r with central (plane) curve $c(s)$. Let $\kappa(s)$ denote the (signed) curvature of c, so that

$$e_1'(s) = \kappa(s)e_2(s), \;\; e_2'(s) = -\kappa(s)e_1(s).$$

We further define the vector fields E_1, E_2 on the parametric surface M by

$$E_1(s,\varphi) = \frac{\frac{\partial\Phi}{\partial s}(s,\varphi)}{\|\frac{\partial\Phi}{\partial s}(s,\varphi)\|}, \quad E_2(s,\varphi) = \frac{\frac{\partial\Phi}{\partial \varphi}(s,\varphi)}{\|\frac{\partial\Phi}{\partial \varphi}(s,\varphi)\|}, \quad N(s,\varphi) = \cos(\varphi)e_2(s) + \sin(\varphi)e_3.$$

(a) Show that E_1, E_2 constitute an orthonormal frame on TM (i.e., an orthonormal basis of T_pM at each point $p = \Phi(s,\varphi)$, and that N is a unit normal vector field to M.

(b) Show that the shape operator S of the tube satisfies at each point of M:

$$S(E_1(s,\varphi)) = \frac{\kappa(s)\cos(\varphi)}{1 - r\kappa(s)\cos(\varphi)}E_1(s,\varphi), \quad S(E_2(s,\varphi)) = -\frac{1}{r}E_2(s,\varphi).$$

(c) What is the Gaussian curvature of the parametric surface at each point $p = \Phi(s,\varphi)$?

(d) Let ∇ denote the Levi-Civita connection on M. Obtain the vector fields:

$$\nabla_{E_1}E_1, \;\; \nabla_{E_1}E_2, \;\; \nabla_{E_2}E_1, \;\; \nabla_{E_2}E_2.$$

6. **Curvature of a 2-torus.** Show that there exists no Riemannian metric on a two-dimensional torus T such that the Gaussian curvature K is everywhere non-zero.

Solution. By the Gauss-Bonnet theorem, given that the Euler characteristic of the torus T is 0, we have

$$\int_T K \,\mathrm{vol}_T = 0$$

where vol_T is the area form on T. But K is a continuous function on T, so it must attain value 0 somewhere.

◇

7. **Holonomy angle.** Let M be a 2-dimensional Riemannian manifold and $c : [0,\ell] \to M$ a smooth simple closed path which is the boundary of a topological disc D. We assume that D is contained in the domain $\mathcal{U}$ of a local orthonormal frame E_1, E_2. Let $V(s)$, $s \in [0,\ell]$, be a parallel unit length vector field along c and $\Theta(s)$ the angle $V(s)$ in the given frame. More precisely, we have

$$V(s) = \cos(\Theta(s))E_1(c(s)) + \sin(\Theta(s))E_2(c(s)).$$

Let ω_{12} be the connection form relative to the given orthonormal frame.

(a) Show that $\Theta'(s) = \omega_{12}(c'(s))$ for each $s \in [0,\ell]$.

(b) Show that

$$\Theta(\ell) - \Theta(0) = \int_D K \,\mathrm{vol}.$$

Thus by parallel translating a vector along a small loop and measuring the change $\Delta\Theta$ in the angle of the vector from the beginning to the end of the transport (Δ indicates change; it is not the Laplacian!), we can measure the curvature of the surface near the loop: $\Delta\Theta \approx \pi r^2 K$, where r is the radius of a very small (approximate) disc.

Solution.

(a) Observe that, since $V(s)$ is parallel, $\frac{d}{ds}\langle E_2, V\rangle = \left\langle \frac{DE_2}{ds}, V\right\rangle$ and

$$\cos(\Theta(s))\Theta'(s) = \frac{d}{ds}\langle E_2, V\rangle = \left\langle \frac{DE_2}{ds}, V\right\rangle = \omega_{12}(c'(s))\langle E_1, V\rangle = \omega_{12}(c'(s))\cos(\Theta(s)).$$

Therefore, $\Theta'(s) = \omega_{12}(c'(s))$.

(b) By integrating over $[0,\ell]$ and using Stokes' Theorem as in the proof of the Gauss-Bonnet formula for polygons,

$$\Theta(\ell) - \Theta(0) = \int_0^\ell \omega_{12}(c'(s))\,ds = \int_D d\omega_{12} = \int_D K \,\mathrm{vol}.$$

$\diamond$

8. **The Pfaffian.** Let $\mathfrak{so}(2m)$ denote the vector space of $2m \times 2m$ skew-symmetric real matrices (the Lie algebra of the special orthogonal group $SO(2m)$). We define a map

$$\mathrm{Pf} : \mathfrak{so}(2m) \to \mathbb{R}$$

as follows. Let $e_1, \ldots, e_{2m}$ be a basis of $\mathbb{R}^{2m}$ and consider for each $A = (a_{ij}) \in \mathfrak{so}(2m)$ the alternating tensor

$$\alpha := \sum_{i<j} a_{ij} e_i \wedge e_j = \frac{1}{2}\sum_{i,j=1}^{2m} a_{ij} e_i \wedge e_j.$$

Then the Pfaffian $\mathrm{Pf}(A)$ is the real number such that

$$\alpha^m = \alpha \wedge \cdots \wedge \alpha = m!\,\mathrm{Pf}(A)\, e_1 \wedge e_2 \wedge \cdots \wedge e_{2m}.$$

(a) Using the definition, show that the Pfaffian of the 4×4 matrix

$$A=\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}$$

is given by

$$\mathrm{Pf}(A)=a_{12}a_{34}-a_{13}a_{24}+a_{23}a_{14}.$$

Observe that the value of the Pfaffian does not depend on the choice of basis of $\mathbb{R}^{2m}$.

(b) Let $J=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and consider the block-diagonal matrix

$$A=\mathrm{diag}\,(a_1 J,\dots,a_m J)\,.$$

Find the associated alternating 2-vector α and show that

$$\mathrm{Pf}(A)=a_1\cdots a_m.$$

(c) If $A\in\mathfrak{so}(2m)$ and B is any $2m\times 2m$ matrix, show that

$$\mathrm{Pf}\left(B^{\top}AB\right)=\mathrm{Pf}(A)\det(B).$$

It follows that the Pfaffian is an $SO(2m)$-invariant polynomial on $\mathfrak{so}(2m)$.

(d) Show that if $A\in\mathfrak{so}(2m)$, then

$$(\mathrm{Pf}(A))^2=\det(A).$$

You may take for granted the following fact from matrix algebra (which I encourage you to try to prove for yourself). There exists an orthogonal matrix $B\in O(2n)$ such that

$$B^{\top}AB=\mathrm{diag}\,(a_1 J,\dots,a_m J)\,.$$

(e) Convince yourself (no need to write it down) that

$$\mathrm{Pf}(A)=\frac{1}{2^m m!}\sum_{\sigma\in S_{2m}}\mathrm{sign}(\sigma)\,a_{\sigma(1)\sigma(2)}\cdots a_{\sigma(2m-1)\sigma(2m)}.$$

9. **Differential forms with coefficients in a vector bundle.** Let M be a smooth manifold and $\pi: E\to M$ a smooth vector bundle. We consider the vector bundle of alternating forms with values in E: $\bigwedge^k(T^*M)\otimes E$ and its smooth sections

$$\Omega^k_M(E):=\Gamma\left(\bigwedge^k(T^*M)\otimes E\right)=\Omega^k(M)\otimes\Gamma(E).$$

Note that the second symbol $\otimes$ is the tensor product of modules. Elements of $\Omega^k_M(E)$ are, locally, linear combinations of terms of the form $\omega\otimes\xi$ where ω is an ordinary smooth k-form on M and ξ is a local section of E. An example of an element of $\Omega^2_M(E)$ is given by the curvature tensor $R(\cdot,\cdot)$ of a connection on a vector bundle F, where $E=F\otimes F^*\cong\mathrm{End}(F)$. We may say that elements of $\Omega_M(E)=\bigoplus\Omega^k_M(E)$ are differential forms with coefficients in $\Gamma(E)$.

Given a connection ∇ on E, we define the *exterior covariant derivative* on $\Omega^*_M(E)$ as the map

$$d^\nabla : \Omega^k_M(E) \to \Omega^{k+1}_M(E)$$

that satisfies

(a) $d^\nabla = \nabla$ for $k = 0$; thus $\left(d^\nabla \xi\right)(X) = \nabla_X \xi$ for $X \in \mathfrak{X}(M)$.

(b) $d^\nabla(\omega \otimes \xi) = d\omega \otimes \xi + (-1)^k \omega \wedge d^\nabla \xi$.

Show the following:

(a) If $\psi \in \Omega^k_M(E)$ and $X_0, \dots, X_k \in \mathfrak{X}(M)$, then

$$\left(d^\nabla \psi\right)(X_0, \dots, X_k) = \sum_i ((-1)^i \nabla_{X_i} \left(\psi\left(X_0, \dots, \widehat{X}_i, \dots, X_k\right)\right) + \sum_{i<j} (-1)^{i+j} \psi\left([X_i, X_j], X_0, \dots, \widehat{X}_j, \dots, \widehat{X}_i, \dots, X_k\right).$$

(b) If $\psi \in \Omega^0_M(E)$ (that is, a smooth section of E), then

$$d^\nabla \circ d^\nabla \xi = R(\cdot, \cdot)\xi$$

where R is the curvature tensor of ∇.

10. **Volume form of a sphere in Cartesian coordinates.** Let $S^{n-1}(\rho)$ be the sphere of radius ρ centered at the origin of $\mathbb{R}^n$, where $\rho(x) = \sqrt{x_1^2 + \cdots + x_n^2}$ and $x_1, \dots, x_n$ are the Cartesian coordinates on $\mathbb{R}^n$. Orient $S^{n-1}(\rho)$ as the boundary of the solid ball of radius ρ.

(a) Prove that the volume form on $S^{n-1}(\rho)$ is (the pull-back to the sphere under the inclusion map of)

$$\mathrm{vol}_{S^{n-1}(\rho)} = \frac{1}{\rho} \sum_{i=1}^n (-1)^{i-1} x^i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

Here the hat $\widehat{dx_i}$ indicates that the term dx_i is absent.

Note: If M is an oriented codimension-1 submanifold of a Riemannian manifold $\tilde{M}$ and N is a unit normal vector to M compatible with the orientation, then the (positive) volume form ω of M is related to the volume form $\tilde{\omega}$ of $\tilde{M}$ according to $\omega = i_N \tilde{\omega}$. Here i_N indicates interior multiplication:

$$i_N \tilde{\omega}(\cdot, \dots, \cdot) = \tilde{\omega}(N, \cdot, \dots, \cdot).$$

(b) Let $\mathrm{vol}_{\mathbb{R}^n} = dx_1 \wedge \cdots \wedge dx_n$ be the standard volume form on $\mathbb{R}^n$. Show that $\mathrm{vol}_{\mathbb{R}^n} = d\rho \wedge \mathrm{vol}_{S^{n-1}(\rho)}$.

(c) Let Z be a vector field on an orientable Riemannian manifold M with volume form ω. The *divergence* of Z, denoted $\mathrm{div} Z$ or $\mathrm{div}_M Z$, is the unique function on M such that

$$\mathcal{L}_Z \omega = (\mathrm{div} Z)\omega.$$

Thus the divergence measures the rate at which the flow of Z changes the volume in M at the linear level. Now suppose Z is a vector field in $\mathbb{R}^n$ such that $Z_x \cdot x = 0$; that is, Z is, at each point, tangent to the sphere containing that point. Show that $\mathrm{div}_{S^{n-1}} Z = \mathrm{div}_{\mathbb{R}^n} Z$. In words: the restriction to the sphere of the divergence of Z regarded as a vector field on $\mathbb{R}^n$ is the divergence of Z regarded as a vector field on the sphere.

(Suggestion: Apply the Lie derivative along Z to $d\rho \wedge \mathrm{vol}_{S^{n-1}}$.) Also check that the divergence of a vector field $Z = \sum_i f_i \partial/\partial x_i$ in $\mathbb{R}^n$ is $\mathrm{div} Z = \sum_i \partial f_i/\partial x_i$.

(d) A skew-symmetric $n \times n$ real matrix z (an element of the Lie algebra $\mathfrak{so}(n)$) induces a vector field Z on $\mathbb{R}^n$ such that at each $x \in \mathbb{R}^n$,

$$Z_x = \frac{d}{dt} e^{tz} x \Big|_{t=0} = zx = \sum_{i,j} z_{ij} x_j \frac{\partial}{\partial x_i}.$$

Show that the restriction of Z to the unit sphere has 0 divergence.

(e) For this final item, recall that if M is an n-dimensional manifold and ω is a smooth n-form with compact support (in particular, if ω is any n-form on a compact M) and $F : M \to M$ is a diffeomorphism then, for every open set $U \subseteq M$,

$$\int_{F(U)} \omega = \int_U F^* \omega.$$

Show that if Z is the vector field on the sphere defined in the previous item and $\Phi_t : S^{n-1} \to S^{n-1}$ is the flow of Z then, for every open set $U \subseteq S^{n-1}$,

$$\mathrm{Vol}(\Phi_t(U)) := \int_{\Phi_t(U)} \mathrm{vol}_{S^{n-1}}$$

is constant in t.

11. **Volume form of an n-sphere in spherical coordinates and hypersurface area.** For $n \geq 3$ we define the spherical coordinates on $\mathbb{R}^n$ as follows. (See Figure **??**.) Let r_k be the distance of the point $(x_1, \ldots, x_k)$ from the origin in $\mathbb{R}^k$:

$$r_k = \sqrt{x_1^2 + \cdots + x_k^2}.$$

The *spherical coordinates* on $\mathbb{R}^2$ are the usual polar coordinates $r = r_2, \theta, 0 \leq \theta < 2\pi$.

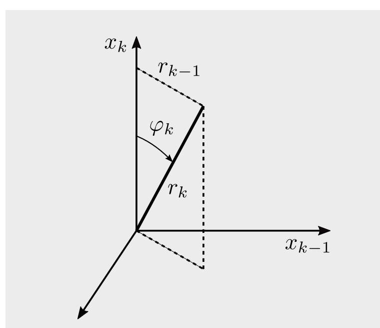

Figure 4: Spherical coordinates.

For $n \geq 3$, if $x = (x_1, \ldots, x_n)$, the angle φ_n is the angle the vector x makes relative to the x_n-axis; it is determined uniquely by the formula

$$\cos \varphi_n = \frac{x_n}{r_n}, \; 0 \leq \varphi_n < \pi.$$

Project x to $\mathbb{R}^{n-1}$ along the x_n-axis. By induction, the spherical coordinates $r_{n-1}, \theta, \varphi_3, \ldots, \varphi_{n-1}$ of the projection $(x_1, \ldots, x_{n-1})$ in $\mathbb{R}^{n-1}$ are defined. Then the spherical coordinates of $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ are defined to be

$r_n, \theta, \varphi_3, \ldots, \varphi_{n-1}, \varphi_n$. Thus for $k = 3, \ldots, n$ we have $\cos\varphi_k = x_k / r_k$. More explicitly, setting $\rho = r_n$,

$$\begin{aligned}
x_n &= \rho \cos\varphi_n \\
x_{n-1} &= \rho \sin\varphi_n \cos\varphi_{n-1} \\
x_{n-2} &= \rho \sin\varphi_n \sin\varphi_{n-1} \cos\varphi_{n-2} \\
&\vdots \\
x_3 &= \rho \sin\varphi_n \cdots \sin\varphi_4 \cos\varphi_3 \\
x_2 &= \rho \sin\varphi_n \cdots \sin\varphi_4 \sin\varphi_3 \sin\theta \\
x_1 &= \rho \sin\varphi_n \cdots \sin\varphi_4 \sin\varphi_3 \cos\theta.
\end{aligned}$$

Note that $r_k = \rho \sin\varphi_n \cdots \sin\varphi_{k+1}$ (or $r_k = \sin\varphi_{k+1} r_{k+1}$) for $k < n$. Setting $\rho = a$, the above defines a parametrization of $S^{n-1}(a)$, which we may write as $x = F(\theta, \varphi_3, \varphi_4, \ldots, \varphi_n)$. (A *parametrization* of a manifold is the inverse of a coordinate map.) Also observe that, we may identify the tangent vectors to the sphere $\partial/\partial\theta$ and $\partial/\partial\varphi_k$ with the vectors $\partial F/\partial\theta$ and $\partial F/\partial\varphi_k$ in $\mathbb{R}^n$.

It is not difficult to show that

$$e_1 = \frac{\partial}{\partial\rho} = \frac{x}{\rho}, \quad e_2 = \frac{1}{r_2}\frac{\partial}{\partial\theta}, \quad e_3 = \frac{1}{r_3}\frac{\partial}{\partial\varphi_3}, \quad \ldots, \quad e_n = \frac{1}{r_n}\frac{\partial}{\partial\varphi_n}$$

constitutes an orthonormal frame along the sphere, where $e_1 = \nu$ is a unit normal vector field.

(a) Give the sphere $S^{n-1}(a)$ the boundary orientation of the closed solid ball of radius a. (This is the orientation defined by the form $\mathrm{vol}_{S^{n-1}(a)}$.) Show that the volume form on $S^{n-1}(a)$ in spherical coordinates is, up to sign,

$$\omega = a^{n-1} \left(\sin^{n-2}\varphi_n\right) \left(\sin^{n-3}\varphi_{n-1}\right) \cdots (\sin\varphi_3)\, d\theta \wedge d\varphi_3 \wedge \cdots \wedge d\varphi_n.$$

Note: A smooth n-form ω on an oriented n-dimensional Riemannian manifold M is the volume form of M if $\omega_p(u_1, \ldots, u_n) = 1$ for any positive orthonormal basis $\{u_j\}$ at p, for all $p \in M$.

(b) By integration by parts, show that

$$\int_0^\pi \sin^n\varphi\, d\varphi = \frac{n-1}{n} \int_0^\pi \sin^{n-2}\varphi\, d\varphi.$$

(c) Give a numerical expression for $\int_0^\pi \sin^{2k}\varphi\, d\varphi$ and $\int_0^\pi \sin^{2k-1}\varphi\, d\varphi$.

(d) Show that

$$\frac{\mathrm{Vol}\left(S^{n+1}(1)\right)}{\mathrm{Vol}\left(S^{n-1}(1)\right)} = \int_0^\pi \sin^n\varphi\, d\varphi \int_0^\pi \sin^{n-1}\varphi\, d\varphi.$$

Although this wasn't asked in the original version of the problem, it is not difficult to conclude that

$$\frac{\mathrm{Vol}\left(S^{n+1}(1)\right)}{\mathrm{Vol}\left(S^{n-1}(1)\right)} = \frac{2\pi}{n}$$

for n both even and odd.

(e) Show that the volume of the $(2n-1)$-dimensional unit sphere is $2\pi^n/(n-1)!$.