Homework set 8 - due 11/01/20

Math 5047 – Renato Feres

Turn in problems 1, 2, 8, 11(b). In problems 8 and 11(b) (and all others), you may freely quote from statements made in other exercises from this assignment.

1. **The gradient, divergence and Laplacian.** Let $(M, \langle\cdot,\cdot\rangle)$ be an oriented Riemannian manifold with volume form ω and $f : M \to \mathbb{R}$ a smooth function. The *gradient* of f is the vector field denoted $q \mapsto \mathrm{grad}_q f$ such that, for all $v \in T_qM$,
$$\langle \mathrm{grad}_q f, v\rangle = df_p(v).$$
Here $df_p v := vf$ is the directional derivative of f along v. The *divergence* of a vector field X on M is the function $q \mapsto \mathrm{div}_q X$ such that
$$\mathscr{L}_X\omega = (\mathrm{div}X)\omega$$
where $\mathscr{L}_X$ is the Lie derivative along X. The *Laplacian* of a smooth function f is
$$\Delta f = \mathrm{div}(\mathrm{grad} f).$$
Let $E_1,\dots,E_n$ be an orthonormal frame on the open set $\mathscr{U} \subseteq M$ and ∇ the Levi-Civita connection on M. Recall that the connection 1-forms relative to the frame $\{E_i\}$ are defined as $\omega_{ij}(v) = \langle E_i, \nabla_v E_j\rangle$.

   (a) Show that $\mathrm{grad} f = \sum_{i=1}^n (E_i f)E_i$ for any smooth function f on $\mathscr{U}$.

   (b) Show that $\mathrm{div}X = \sum_{i=1}^n E_i\langle E_i, X\rangle + \sum_{i,j=1}^n \langle E_j, X\rangle\omega_{ij}(E_i)$ for any smooth vector field X on $\mathscr{U}$.

   (c) Show that $\Delta f = \sum_{i=1}^n E_iE_i f + \sum_{i,j=1}^n \omega_{ij}(E_i)E_j f$ for any smooth function f on $\mathscr{U}$.

   (d) Show that if f, g are smooth functions on M,
$$\Delta(fg) = f\Delta g + g\Delta f + 2\langle \mathrm{grad} f, \mathrm{grad} g\rangle.$$

   (e) We say that the orthonormal frame $\{E_i\}$ is *geodesic* at $q \in \mathscr{U}$ if $(\nabla_{E_i}E_j)_q = 0$ for all i, j. Convince yourself that a geodesic frame at q exists on some neighborhood of q, but you don't have to write it down. (The key idea is this: Let $\mathscr{U}$ be the image under Exp_q of a small ball in T_qM centered at the origin. Let $e_1,\dots,e_n$ be an orthonormal basis of T_qM. Define E_i at $p \in \mathscr{U}$ as the parallel transport of e_i along the unique geodesic from q to p contained in $\mathscr{U}$.) Conclude that, relative to a geodesic orthonormal frame $\{E_i\}$ at q on a sufficiently small neighborhood $\mathscr{U}$ of q, the divergence of X and the Laplacian of f at q (but not necessarily at other points of $\mathscr{U}$) are
$$\mathrm{div}_q X = \sum_{i=1}^n E_i(\langle E_i, X\rangle)(q), \quad (\Delta f)(q) = \sum_{i=1}^n E_i(E_i f)(q).$$

2. **Harmonic functions are constant on a compact manifold.** For this problem, you may need to review Stokes' theorem on a manifold with boundary:
$$\int_{\partial M} \nu = \int_M d\nu$$

where v is a compactly supported $(n-1)$-form on the n-dimensional manifold M.

(a) Show that if X is a (compactly supported) smooth vector field on the oriented Riemannian manifold M, then

$$\int_M (\mathrm{div} X)\mathrm{vol}_M = \int_{\partial M} \langle N, X\rangle\, \mathrm{vol}_{\partial M}$$

where vol_M is the volume form of M and $\mathrm{vol}_{\partial M}$ is the induced volume form on ∂M. This is the *divergence theorem*. In words: the integral of the divergence of the vector field X is the flux of that vector field across the boundary.

(b) Using the divergence theorem, show that if f is a smooth function on a compact Riemannian manifold M without boundary, then

$$\int_M (\Delta f)\mathrm{vol}_M = 0.$$

(This is the statement that the volume form is harmonic in the weak sense.)

(c) Suppose f is a harmonic function on the compact, connected Riemannian manifold M without boundary; that is, $\Delta f = 0$. Show that f must be constant. (Suggestion: Start with $0 = \int_M \Delta\left(f^2\right)\mathrm{vol}_M$ and express the integral in terms of the gradient of f using an identity from the previous problem.)

3. **Geodesics in lens spaces.** Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ by letting (x_1, x_2, x_3, x_4) correspond to $(x_1 + ix_2, x_3 + ix_4)$. Let

$$S^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\right\},$$

and let $h : S^3 \to S^3$ be given by

$$h(z_1, z_2) = \left(e^{\frac{2\pi i}{q}} z_1, e^{\frac{2\pi i r}{q}} z_2\right), \quad (z_1, z_2) \in S^3,$$

where q and r are relatively prime integers, $q > 2$.

(a) Show that $G = \left\{\mathrm{id}, h, \cdots, h^{q-1}\right\}$ is a group of isometries of the sphere S^3, with the usual metric, which operates in a totally discontinuous manner. The manifold S^3/G is called a *lens space*.

(b) Consider S^3/G with the metric induced by the projection $p : S^3 \to S^3/G$. Show that all the geodesics of S^3/G are closed but can have different lengths.

4. **Exterior derivative expressed in terms of a connection.** If ω is a differential k-form on a smooth manifold M equipped with a torsion-free (i.e., symmetric) connection ∇, show that

$$d\omega(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i \left(\nabla_{X_i}\omega\right)\left(X_0, \ldots, \widehat{X}_i, \ldots, X_k\right),$$

where $X_0, X_1, \ldots, X_k \in \mathfrak{X}(M)$ and the hat symbol $\widehat{X}$ indicates that the corresponding vector field is dropped.

Note: A useful identity involving the exterior product which, unfortunately, is not in the appendix of Lee's text, is the following:

$$\begin{aligned}(d\omega)(X_0, \ldots, X_k) = &\sum_{i=0}^{k} (-1)^i X_i\left(\omega\left(X_0, \ldots, \widehat{X}_i, \ldots, X_k\right)\right) \\ &+ \sum_{i<j} (-1)^{i+j} \omega\left([X_i, X_j], X_0, \ldots, \widehat{X}_j, \ldots, \widehat{X}_i, \ldots, X_k\right).\end{aligned}$$

You may take it for granted.

5. **Riemannian metric on the tangent bundle of a Riemannian manifold.** It is possible to introduce a Riemannian metric on the tangent bundle $N = TM$ of a Riemannian manifold M in the following manner. Let $(p, v) \in N$ and V, W be tangent vectors in N at (p, v). Choose curves $\alpha : t \mapsto (p(t), v(t))$ and $\beta : s \mapsto (q(s), w(s))$ in N with $p(0) = q(0) = p$, $v(0) = w(0) = v$, and $V = \alpha'(0)$, $W = \beta'(0)$. Define an inner product on N by

$$\langle V, W\rangle_{(p,v)} := \left\langle d\pi_{(p,v)} V, d\pi_{(p,v)} W\right\rangle_p + \left\langle \frac{Dv}{dt}(0), \frac{Dw}{ds}(0)\right\rangle_p, \tag{1}$$

where $d\pi_{(p,v)}$ is the differential of the base point projection $\pi : N \to M$. It is not difficult to show that this inner product is well-defined (i.e., it is independent of the choices of α and β) and that it indeed defines a Riemannian metric on N. This is often called the *Sasaki* metric on TM.

A vector V at $(p, v) \in N$ that is orthogonal (in the metric (1)) to the fiber $\pi^{-1}(p) \cong T_pM$ is called a *horizontal vector*. A curve $t \mapsto (p(t), v(t))$ in N is *horizontal* if its tangent vector is horizontal for all t. Here are some facts we know (from other homework assignments or from class):

- The curve $t \mapsto (p(t), v(t))$ is horizontal if and only if the vector field $v(t)$ is parallel along $p(t) \in M$.
- We define the *geodesic vector field* Z as the horizontal vector field on N such that $d\pi_{(p,v)}Z(p, v) = v$ for all $(p, v) \in N$. The flow line $t \mapsto \Phi_t(p, v)$ of the flow Φ_t of Z projects under π to the geodesic on M with initial conditions (p, v).

Problems:

(a) Let V be a vertical vector in $T_{(p,v)}N$. Recall that we may regard V as a vector in T_pM due to the canonical isomorphism between T_pM and the vertical subspace of $T_{(p,v)}N$:

$$V \in T_pM \mapsto \frac{d}{dt}(v + tV)\Big|_{t=0} \in T_{(p,v)}N.$$

Show that $J(t) := d(\pi \circ \Phi_t)_{(p,v)}V$ is a Jacobi field with initial conditions $J(0) = 0$ and $\frac{DJ}{dt}(0) = V$.

(b) Let V be a horizontal vector in $T_{(p,v)}N$ and $w := d\pi_{(p,v)}V$. Show that $J(t) := d(\pi \circ \Phi_t)_{(p,v)}V$ is a Jacobi field with initial conditions $J(0) = w$ and $\frac{DJ}{dt}(0) = 0$.

6. **Riemannian submersions**. Read about Riemannian products (Chapter 2, page 20) and Riemannian submersions (Chapter 2, page 21) in Lee's text. Let $\overline{M}$ and M be manifolds of dimension $n + k$ and n, respectively. A differentiable mapping $f : \overline{M} \to M$ is called *submersion* if f is surjective, and for all $\overline{p} \in \overline{M}$, the differential $df_{\overline{p}} : T_{\overline{p}}\overline{M} \to T_{f(\overline{p})}M$ has rank n. In this case, for all $p \in M$, the *fiber* $f^{-1}(p) = F_p$ is a submanifold of $\overline{M}$ (by the implicit function theorem) and a tangent vector of $\overline{M}$, tangent to some F_p, $p \in M$, is called a *vertical vector* of the submersion. If, in addition, $\overline{M}$ and M have Riemannian metrics, the submersion f is said to be *Riemannian* if $df_p : T_p\overline{M} \to T_{f(p)}M$ preserves lengths of vectors orthogonal to F_p, for all $p \in \overline{M}$.

(a) If $M_1 \times M_2$ is the Riemannian product, then the natural projections $\pi_i : M_1 \times M_2 \to M_i$, $i = 1, 2$, are Riemannian submersions.

(b) If the tangent bundle $N = TM$ is given the Riemannian metric as in the previous exercise, then the projection $\pi : TM \to M$ is a Riemannian submersion.

7. **Connection of a Riemannian submersion.** Let $f : \overline{M} \to M$ be a Riemannian submersion. A vector $\overline{x} \in T_{\overline{p}}\overline{M}$ is *horizontal* if it is orthogonal to the fiber. The tangent space $T_{\overline{p}}\overline{M}$ then admits a decomposition

$$T_{\overline{p}}\overline{M} = \left(T_{\overline{p}}\overline{M}\right)^h \oplus \left(T_{\overline{p}}\overline{M}\right)^v,$$

where $\left(T_{\overline{p}}\overline{M}\right)^h$ and $\left(T_{\overline{p}}\overline{M}\right)^v$ denote the subspaces of horizontal and vertical vectors, respectively. If $X \in \mathfrak{X}(M)$, the *horizontal lift* $\overline{X}$ of X is the horizontal field defined by $df_{\overline{p}}\left(\overline{X}(\overline{p})\right) = X(f(p))$. You may take for granted (but think about it!) that $\overline{X}$ is a smooth vector field.

(a) Show that $\overline{X}$ is a smooth vector field.

(b) Let ∇ and $\overline{\nabla}$ be the Riemannian connections of M and $\overline{M}$, respectively. Show that

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla_X Y} + \frac{1}{2}\left[\overline{X}, \overline{Y}\right]^v, \quad X, Y \in \mathfrak{X}(M),$$

where Z^v is the vertical component of Z.

(c) $\left[\overline{X}, \overline{Y}\right]^v(\overline{p})$ depends only on $\overline{X}(\overline{p})$ and $\overline{Y}(\overline{p})$.

Hint for (b): Let $X, Y, Z \in \mathfrak{X}(M)$. Let $T \in \mathfrak{X}\left(\overline{M}\right)$ be a vertical field. Observe that

$$\left\langle \overline{X}, T\right\rangle = \left\langle \overline{Y}, T\right\rangle = \left\langle \overline{Z}, T\right\rangle = 0, \;\; \overline{X}\left\langle \overline{Y}, \overline{Z}\right\rangle = X\langle Y, Z\rangle.$$

Also

$$df\left[\overline{X}, T\right] = 0, \quad [X, Y] = \left[df\overline{X}, df\overline{Y}\right] = df\left[\overline{X}, \overline{Y}\right]$$

and

$$T\left\langle \overline{X}, \overline{Y}\right\rangle = 0.$$

From this we can conclude that

$$\left\langle \left[\overline{X}, \overline{Y}\right], \overline{Z}\right\rangle = \langle [X, Y], Z\rangle, \quad \left\langle \left[\overline{X}, T\right], \overline{Y}\right\rangle = 0.$$

We can now use the expression for the Levi-Civita connection in Theorem 5.10 to obtain

$$\left\langle \overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}\right\rangle = \langle \nabla_X Y, Z\rangle, \;\; 2\left\langle \overline{\nabla}_{\overline{X}}\overline{Y}, T\right\rangle = \left\langle T, \left[\overline{X}, \overline{Y}\right]\right\rangle.$$

This implies the desired identities.

8. **Curvature of a Riemannian submersion.** Let $f : \overline{M} \to M$ be a Riemannian submersion. Let $X, Y, X, W \in \mathfrak{X}(M)$ and $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ be their horizontal lifts. Let R and $\overline{R}$ be the curvature tensors of M and $\overline{M}$, respectively. Prove that

(a)

$$\langle R(X,Y)Z, W\rangle \circ f = \left\langle \overline{R}(\overline{X}, \overline{Y})\overline{Z}, \overline{W}\right\rangle - \frac{1}{4}\left\langle \left[\overline{X}, \overline{Z}\right]^v, \left[\overline{Y}, \overline{W}\right]^v\right\rangle + \frac{1}{4}\left\langle \left[\overline{Y}, \overline{Z}\right]^v, \left[\overline{X}, \overline{W}\right]^v\right\rangle - \frac{1}{2}\left\langle \left[\overline{Z}, \overline{W}\right]^v, \left[\overline{X}, \overline{Y}\right]^v\right\rangle.$$

(b) If σ is the plane generated by the orthonormal vectors $X, Y \in \mathfrak{X}(M)$ and $\overline{\sigma}$ is the plane generated by $\overline{X}$, $\overline{Y}$, then

$$K(\sigma) = \overline{K}(\sigma) + \frac{3}{4}\left\| \left[\overline{X}, \overline{Y}\right]^v \right\|^2 \geq \overline{K}(\overline{\sigma}).$$

9. **The Hopf bundle and $\mathbb{C}P^1$.** (This is only reading.) The complex projective space of complex dimension 1 is defined as the set of complex lines (one-dimensional complex vector subspaces) in $\mathbb{C}^2$. Formally, it is defined as the quotient of $\mathbb{C}^2 \setminus \{0\}$ (the 2-dimensional complex vector space minus the origin) under the equivalence relation that identifies any two non-zero vectors that are collinear. Equivalently, let S^3 denote the 3-dimensional

unit sphere, regarded as the submanifold of $\mathbb{C}^2$ consisting of pairs (z_1, z_2) such that $|z_1|^2 + |z_2|^2 = 1$, and define on it the action by the group $U(1)$ of unit modulus complex numbers:

$$\left(e^{i\theta}, (z_1, z_2)\right) \mapsto \left(e^{i\theta} z_1, e^{i\theta} z_2\right).$$

Then $\mathbb{C}P^1$ is the quotient of S^3 under this action. Notice that the orbits are circles, so we should expect the quotient to be a 2-dimensional (real) manifold. In fact, $\mathbb{C}P^1$ has the structure of a smooth manifold diffeomorphic to S^2. (It is also a 1-dimensional complex manifold.) The quotient map defines a *circle bundle* $\pi : S^3 \to S^2$ called the *Hopf bundle*.

There are several ways to show that $\mathbb{C}P^1$ is diffeomorphic to S^2. I find the following, inspired by a view from quantum theory, especially interesting. Let M denote the space of self-adjoint 2×2 complex matrices of trace 0. Such matrices can be written as

$$\sigma \cdot x := x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 = \begin{pmatrix} x_3 & x_1 + i x_2 \\ x_1 - i x_2 & -x_3 \end{pmatrix}$$

where

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the so called *Pauli matrices*. The x_j are real. We are interested in $x \in S^2$. Note that

$$(\sigma \cdot x)^2 = |x|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

where $|x|^2 = x_1^2 + x_2^2 + x_3^2 = 1$ and I is the identity matrix. The matrices $\sigma \cdot x$ for $x \in S^2$ have eigenvalues ± 1. In fact, consider the matrices $P_\pm(x) = \frac{1}{2}(I \pm \sigma \cdot x)$. They are the orthogonal projections to the 1-dimensional subspaces of $\mathbb{C}^2$ spanned by the eigenvalues ± 1. This is a consequence of the easy to verify matrix identities (where P^* indicates the transpose-conjugate, or matrix adjoint):

$$P_\pm(x)^* = P_\pm(x), \;\; P_\pm(x)^2 = P_\pm(x), \;\; P_-(x)P_+(x) = P_+(x)P_-(x) = 0, \;\; P_-(x) + P_+(x) = I, \;\; \sigma \cdot x P_\pm(x) = \pm P_\pm(x).$$

We have in this way used S^2 to parametrize all the (both) self-adjoint and unitary 2×2 complex matrices having distinct eigenvalues. Let $u(x)$ be a unit length eigenvector in $\mathbb{C}^2$ associated to the eigenvalue 1 of $P_+(x)$. Thus $u(x) \in S^3$ and $P_+(x)u(x) = u(x)$. Observe that $P_+(x)$ is the rank-1 orthogonal projection matrix given by

$$v \mapsto P_+(x)v = \langle u(x), v\rangle u(x)$$

for $v \in \mathbb{C}^2$. Here $\langle \cdot, \cdot \rangle$ is the Hermitian inner product in $\mathbb{C}^2$ given by: $\langle (z_1, z_2), (w_1, w_2)\rangle = \overline{z}_1 w_1 + \overline{z}_2 w_2$.

We may write $P_+(x) = u(x) \otimes u(x)^*$, where $u(x)^*$ is the dual vector to $u(x)$. (In quantum theory notation, using Dirac's bra-kets, $u(x) \otimes u(x)^* = |x\rangle\langle x|$.)

The outcome of the above discussion is this: if we identify each element of projective space $\mathbb{C}P^1$ with the orthogonal projection operator $P_+(x)$ in the Hilbert space $\mathbb{C}^2$ onto the eigenspace of $\sigma \cdot x$ for eigenvalue 1, we obtain a map $S^2 \to \mathbb{C}P^1$ which it is not difficult to show is a diffeomorphism. (In this way, the complex projective space can be regarded as a submanifold of a three-dimensional (over $\mathbb{R}$) space of matrices.) Note that this map is well-defined since a different choice $e^{i\theta} u(x)$ of unit length eigenvector will give the same projection map

$u(x)\otimes u(x)^*$. On the other hand, each projection can be written as the image of a point in S^3 under the map

$$\pi : S^3 = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\} \to \mathbb{C}P^1 \cong S^2$$

such that $\pi(z) = z \otimes z^*$. Observe that $\pi\left(e^{\lambda i} z\right) = \pi(z)$ since

$$\pi\left(e^{\lambda i} z\right) = \left(e^{\lambda i} z\right) \otimes \left(e^{\lambda i} z\right)^* = e^{\lambda i} e^{-\lambda i} z \otimes z^* = z \otimes z^* = \pi(z).$$

So we have, as expected, that $\mathbb{C}P^1$ is a quotient of S^3 under the diagonal action of the circle group $U(1)$ on S^3:

$$e^{\lambda i}(z_1, z_2) = \left(e^{\lambda i} z_1, e^{\lambda i} z_2\right).$$

The identification between S^2 and $\mathbb{C}P^1$ is accomplished by the bijection $x \mapsto \sigma \cdot x$. The resulting map $\pi : S^3 \to S^2$ is, again, the Hopf bundle.

⋄

10. **The complex projective space $\mathbb{C}P^n$.** On $\mathbb{C}^{n+1} \setminus \{0\} = \left\{Z = (z_0, \ldots, z_n) \neq 0 : z_j = x_j + i y_j\right\}$ define the equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$: $Z = (z_0, \ldots, z_n) \sim W = (w_0, \ldots, w_n)$ if $z_j = \lambda w_j$ for all j where $\lambda \in \mathbb{C} \setminus \{0\}$. The equivalence class represented by Z—the complex line through the origin containing Z—will be denoted $[Z]$. The quotient

$$\mathbb{C}P^n := \mathbb{C}^{n+1} \setminus \{0\} / \sim$$

is called the *complex projective space* of complex dimension n.

(a) Show that $\mathbb{C}P^n$ has a differentiable structure of a manifold of real dimension $2n$. Note: The proof is in Example C.19, page 413, of Lee's text.

(b) Let $\langle Z, W\rangle = \overline{z}_0 w_0 + \cdots + \overline{z}_n w_n$ be the Hermitian product on $\mathbb{C}^{n+1}$, where the bar denotes complex conjugation. Identify $\mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}$ by writing $z_j = x_j + i y_j = (x_j, y_j)$. Show that

$$S^{2n+1} = \left\{N \in \mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2} : \langle N, N\rangle = 1\right\}$$

is the unit sphere in $\mathbb{R}^{2n+2}$.

(c) Show that $\sim$ induces on S^{2n+1} the following equivalence relation: $Z \sim W$ if $W = e^{i\theta} Z$. Establish that there exists a differentiable map (the Hopf bundle) $f : S^{2n+1} \to \mathbb{C}P^n$, such that

$$f^{-1}([Z]) = \left\{e^{i\theta} N \in S^{2n+1} : N \in [Z] \cap S^{2n+1}, 0 \le \theta \le 2\pi\right\} = [Z] \cap S^{2n+1}.$$

(d) Show that f is a submersion. (This is already in Example C.19, so nothing more to do here.)

11. **Curvature of the complex projective space.** (This is similar to Problem 8-13, page 258 of Lee's text.) Define a Riemannian metric on $\mathbb{C}^{n+1} \setminus \{0\}$ in the following way: If $Z \in \mathbb{C}^{n+1} \setminus \{0\}$ and $V, W \in T_Z(\mathbb{C}^{n+1} \setminus \{0\})$,

$$\langle V, W\rangle_Z = \frac{\mathrm{Real}(\langle V, W\rangle_0)}{\langle Z, Z\rangle_0}$$

where $\langle V, W\rangle_0 := \overline{v}_0 w_0 + \cdots + \overline{v}_n w_n$. Observe that the metric $\langle \cdot, \cdot\rangle$ restricted to $S^{2n+1} \subseteq \mathbb{C}^{n+1} \setminus \{0\}$ coincides with the metric induced from $\mathbb{R}^{2n+2}$.

(a) Show that, for all $0 \le \theta \le 2\pi$, $e^{i\theta} : S^{2n+1} \to S^{2n+1}$ is an isometry, and that, therefore, it is possible to define a Riemannian metric on $P^n(\mathbb{C})$ in such a way that the submersion f defined in the previous problem is Riemannian.

(b) Show that, in this metric, the sectional curvature of $P^n(\mathbb{C})$ is given by

$$K(\sigma) = 1 + 3\cos^2\varphi,$$

where σ is generated by the orthonormal pair X, Y, $\cos\varphi = \left\langle \overline{X}, i\overline{Y} \right\rangle$, and $\overline{X}, \overline{Y}$ are the horizontal lifts of X and Y, respectively. In particular, $1 \le K(\sigma) \le 4$.

Hint for part (b): Let Z be the position vector describing S^{2n+1}. Since

$$\left(\frac{d}{d\theta} e^{i\theta} Z \right)_{\theta=0} = iZ$$

$iZ \in T_Z S^{2n+1}$ and it is vertical. Let $\overline{\nabla}$ be the Levi-Civita connection of $\mathbb{R}^{2n+2} \approx \mathbb{C}^{n+1}$ and $X, Y \in \mathfrak{X}(\mathbb{C}P^n)$. Take $\alpha : (-\epsilon, \epsilon) \to S^{2n+1}$ with $\alpha(0) = Z$, $\alpha'(0) = \overline{X}$. Then

$$\left(\overline{\nabla}_{\overline{X}} iZ\right)_Z = \frac{d}{dt} iZ \circ \alpha(t) \bigg|_{t=0} = i\alpha(t)|_{t=0} = i\alpha'(0) = i\overline{X}.$$

Therefore

$$\left\langle \left[\overline{X}, \overline{Y}\right], iZ \right\rangle = \left\langle \overline{\nabla}_{\overline{X}} \overline{Y} - \overline{\nabla}_{\overline{Y}} \overline{X}, iZ \right\rangle = -\left\langle i\overline{X}, \overline{Y} \right\rangle + \left\langle i\overline{Y}, \overline{X} \right\rangle = 2\cos\varphi.$$

Now use the above facts about the sectional curvature of a Riemannian submersion.