Homework set 7 - due 10/25/20

Math 5047 – Renato Feres

Turn in problems 2, 7, 8, 9.

1. **The sectional curvatures determine the curvature tensor.** For this exercise, first read Proposition 8.31 (and its proof), which states that the sectional curvatures completely specify the full curvature tensor R. Without using this fact, but using the idea of the proof, which relies on the symmetries of the curvature tensor given in a previous homework (see also Proposition 7.12, page 202 of Lee's text), show that if the sectional curvatures are equal to 0 then $R = 0$.

2. **Zero sectional curvature implies** exp **is an isometry.** Let M be a Riemannian manifold with sectional curvature identically zero. Show that, for every $p \in M$, the mapping $\exp_p : B_\epsilon(0) \subseteq T_pM \to B_\epsilon(p)$ is an isometry, where $B_\epsilon(p)$ is a normal ball at p.

3. **Useful Identity involving the curvature tensor.** Let V be a smooth vector field on a Riemannian manifold M and $x_1, \dots, x_n$ local coordinates. Show that

$$\frac{D}{\partial x_i}\frac{D}{\partial x_j}V = \frac{D}{\partial x_j}\frac{D}{\partial x_i}V + R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)V, \tag{1}$$

where R is the Riemann curvature tensor. The proof should be fairly quick. Now Read Proposition 7.5 (and its proof), page 197 of Lee's text. Make sure you understand in what way this proposition is more general than the result you are asked to prove, and so necessitates a longer proof.

4. **Path independent parallel transport implies zero curvature.** (do Carmo's Riemannian Geometry.) Let M be a Riemannian manifold with the following property: given any two points $p, q \in M$, the parallel transport from p to q does not depend on the curve joining p and q. Prove that the curvature of M is identically zero, that is, $R(X,Y)Z = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$.

   Hint: Consider a parametrized surface $f : U \subseteq \mathbb{R}^2 \to M$ where

$$U = \{(s,t) \in \mathbb{R}^2 : -\epsilon < t < 1+\epsilon, -\epsilon < s < 1+\epsilon, \epsilon > 0\}.$$

   and $f(s,0) = f(0,0)$, for all s. Let $V_0 \in T_{f(0,0)}M$ and define a field V along f by : $V(s,0) = V_0$ and, if $t \neq 0$, $V(s,t)$ is the parallel transport of V_0 along the curve $t \to f(s,t)$. Then, from Proposition 7.5 (see the previous problem),

$$\frac{D}{\partial s}\frac{D}{\partial t}V = 0 = \frac{D}{\partial t}\frac{D}{\partial s}V + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)V.$$

   Since parallel transport does not depend on the curve chosen, $V(s,1)$ is the parallel transport of $V(0,1)$ along the curve $s \mapsto f(s,1)$, hence $\frac{D}{\partial s}V(s,1) = 0$. Thus

$$R_{f(0,1)}\left(\frac{\partial f}{\partial t}(0,1), \frac{\partial f}{\partial s}(0,1)\right)V(0,1) = 0.$$

Now use the arbitrariness of f and V_0 to draw the desired conclusion about R.

5. **Converse of the previous item.** If the curvature tensor of a torsion-free connection (not necessarily Riemannian) is zero, then parallel transport does not depend on the curve joining two points on simply connected regions. In this case its is possible to define local coordinates whose coordinate vector fields are parallel. This characterizes the manifold as being locally affinely isomorphic to Euclidean space.

   A finer result for Riemannian manifolds is discussed in Chapter 7 of Lee's text. We say that the Riemannian manifold M is *flat* if it is locally isometric to a Euclidean space (page 195). Read the discussion on pages 199-201, specifically Theorem 7.10, which states that a Riemannian manifold is flat if and only if its curvature tensor vanishes. I'll return to this in class if we find time.

6. **Geodesic variation from Jacobi field.** (From do Carmo's text.) Let M be a Riemannian manifold, $\gamma : [0,1] \to M$ a geodesic, and J a Jacobi field along γ. Prove that there exists a parametrized surface $f(t,s)$, where $f(t,0) = \gamma(t)$ and the curves $t \to f(t,s)$ are geodesics, such that $J(t) = \frac{\partial f}{\partial s}(t,0)$.

   Hint: Choose a curve $\lambda(s)$, $s \in (-\epsilon, \epsilon)$, in M such that $\lambda(0) = \gamma(0)$, $\lambda'(0) = J(0)$. Along λ choose a vector field $W(s)$ with $W(0) = \gamma'(0)$ and $\frac{DW}{ds}(0) = \frac{DJ}{dt}(0)$. Define $f(s,t) = \exp_{\lambda(s)} tW(s)$ and verify that $\frac{\partial f}{\partial s}(0,0) = \frac{d\lambda}{ds}(0) = J(0)$ and

$$\frac{D}{dt}\frac{\partial f}{\partial s}(0,0) = \frac{D}{ds}\frac{\partial f}{\partial t}(0,0) = \frac{DW}{ds}(0) = \frac{DJ}{dt}(0).$$

7. **Conjugate locus.** Let M be a Riamnnian manifold and let $\gamma : [0,a] \to M$ be a geodesic. The point $\gamma(t_0)$ is said to be *conjugate* to $\gamma(0)$ along γ, $t_0 \in (0,a]$, if there exists a Jacobi field J along γ, not identitcally zero, with $J(0) = 0 = J(t_0)$. The maximum number of such linearly independent fields is called the *multiplicity* of the conjugate point $\gamma(t_0)$. The set of (first) conjugate points to $p \in M$ is called the *conjugate locus* of p and denoted $C(p)$. Read about conjugate points in Lee's text, pages 297-299, in particular Proposition 10.20 (page 299), which relates conjugate points with critical points of the exponential map.

   Now suppose that the Riemannian manifold M has non-positive sectional curvature and let $\gamma : [0,a] \to M$ be a geodesic. Prove that, for all p, the conjugate locus $C(p)$ is empty.

   Hint: Assume the existence of a non-trivial Jacobi field along the geodesic $\gamma : [0,a] \to M$, with $\gamma(0) = p$, $J(0) = J(a) = 0$. Use the Jacobi equation to show that

$$\frac{d}{dt}\left\langle \frac{D}{dt}J, J \right\rangle \geq 0.$$

   Conclude that $\left\langle \frac{D}{dt}J, J \right\rangle$ is identically equal to 0. Since

$$\frac{d}{dt}\langle J, J\rangle = 2\left\langle \frac{D}{dt}J, J \right\rangle = 0$$

   for all t we arrive at the contradiction that $\|J\|^2$ is constant.

8. **Locally symmetric spaces.** Read about locally symmetric spaces in Lee's text, pages 295-297, in particular Theorem 10.19. Here we will take a short-cut (justified by that theorem) and say that a Riemannian manifold M is a *locally symmetric space* if the the curvature tensor R is parallel: $\nabla R = 0$.

   (a) Let M be a locally symmetric space and let $\gamma : [0,\ell) \to M$ be a smooth curve in M. Let X, Y, Z be parallel vector fields along γ. Prove that $R(X,Y)Z$ is a parallel vector field along γ.

   (b) Prove that if M is locally symmetric, connected, and has dimension two, then M has constant (sectional) curvature.

(c) Prove that if M has constant (sectional) curvature, then M is a locally symmetric space.

9. **Jacobi fields and conjugate points on locally symmetric spaces** (From do Carmo's text.) Let $\gamma : [0,\infty) \to M$ be a geodesic in a locally symmetric space M and let $v = \gamma'(0)$ be its velocity at $p = \gamma(0)$. Define a linear transformation $K_v : T_pM \to T_pM$ by

$$K_v(x) = R(x, v)v, \quad x \in T_pM.$$

(a) Prove that K_v is self-adjoint.

(b) Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM that diagonalizes K_v; that is,

$$K_v(e_i) = \lambda_i e_i, \quad i = 1, \ldots, n.$$

Extend the e_i to fields along γ by parallel transport. Show that, for all t,

$$K_{\gamma'(t)}(e_i(t)) = \lambda_i e_i(t),$$

where λ_i does not depend on t.

(c) Let $J(t) = \sum_i x_i(t)e_i(t)$ be a Jacobi field along γ. Show that the Jacobi equation is equivalent the the system

$$\frac{d^2 x_i}{dt^2} + \lambda_i x_i = 0, \quad i = 1, \ldots, n.$$

(d) Show that the conjugate points of p along γ are given by $\gamma\left(\pi k/\sqrt{\lambda_i}\right)$, where k is a positive integer and λ_i is a positive eigenvalue of K_v.