Homework set 6 - due 10/18/20

Math 5047 – Renato Feres

Turn in problems 1, 2, 5(a,b), 7.

1. **Naturality of the exponential map.** Let $f : M \to N$ be an isometry of Riemannian manifolds, and $p \in M$. Suppose $U \subseteq T_pM$ and $V \subseteq T_{f(p)}N$ are neighborhoods of the origin on which the exponential maps $\exp_p$ and $\exp_{f(p)}$ are defined. If the differential df_p maps U into V, show that

$$\exp_{f(p)} \circ df_p = f \circ \exp_p.$$

   Note: Convince yourself of the following fact (you don't need to write down the proof for this): a Riemannian isometry maps (parametrized) geodesics to geodesics. This is an easy fact once you verify that the pull-back of the Levi-Civita connection on N under a Riemannian isometry is the Levi-Civita connection on N.

2. **Isometries of the unit sphere.** Let $S^{n-1} = S^{n-1}(1)$ be the sphere of radius 1 centered at the origin of $\mathbb{R}^n$. We give S^{n-1} the Riemannian metric that makes the inclusion map an isometric embedding. This means that the metric on the sphere is the restriction to its tangent spaces of the standard dot product: $\langle u, v\rangle_x = u \cdot v$. Note the natural identification

$$T_xS^{n-1} = \{u \in \mathbb{R}^n : x \cdot u = 0\}.$$

   It is not difficult to show that the restriction to the sphere of an orthogonal transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is a Riemannian isometry. Conversely, show that every isometry $f : S^{n-1} \to S^{n-1}$ is the restriction to S^{n-1} of an orthogonal transformation. Thus the isometry group of the sphere is the orthogonal group

$$O(n) = \{A : \text{ an } n \times n \text{ real matrix such that } A^\intercal A = I\}.$$

   Suggestion: By composing f with an orthogonal map, we can assume that there is a point $x \in S^{n-1}$ such that $f(x) = x$ and df_x is the identity map on the tangent space at x. Now use the result of the previous exercise to argue that f must be the identity map on a neighborhood of x and, consequently, (why?) the identity map.

3. **Triangle inequality for the distance function on a Riemannian manifold.** On the connected Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ we define the norm of a vector $v \in T_pM$ as $\|v\| = \sqrt{\langle v, v\rangle}$ and the length of a smooth curve $\gamma : [a,b] \to M$ as $\ell(\gamma) = \int_a^b \|\gamma'(t)\|\, dt$. If the continuous curve is only piecewise smooth, we define its length as the sum of the lengths of the smooth pieces.

   Given two points p, q on M, we define the distance between them as

$$d(p.q) = \inf_\gamma \ell(\gamma),$$

   where the infimum is taken over all piecewise smooth curves joining p to q.

(a) Show that the arclength of a curve γ does not depend on its parametrization. That is, if

$$s \in [c,d] \mapsto t(s) \in [a,b]$$

is a smooth increasing function and $\eta(s) = \gamma(t(s))$ then

$$\ell(\eta) = \int_c^d \|\eta'(s)\|\, ds = \int_a^b \|\gamma'(t)\|\, dt = \ell(\gamma).$$

(b) Prove the triangle inequality for the distance function on a Riemannian manifold M: for all $p, q, r \in M$,

$$d(p,r) \le d(p,q) + d(q,r).$$

4. **Geodesics and distance. Chapter 6 of Lee's text.** Read the following sections of Lee's text:

(a) The definition of a one-parameter family of curves $\Gamma(s,t)$ (page 152 of Lee's text). Here s, t are real parameters so that, for each s, $t \mapsto \Gamma(s,t)$ are called the *main curves* and, for each t, $s \mapsto \Gamma(s,t)$ are called the *transverse curves*. Here Γ is allowed to be piecewise smooth. See Figure 6.1 on page 153. The context for introducing one-parameter families of curves is the following: we begin with a continuous and piecewise smooth curve $\gamma(t)$ and wish to define a variation of it (in the sense of calculus of variations; more on this in the next problem). Then $\gamma(t) = \Gamma(0,t)$ is the main curve and $\gamma_s(t) = \Gamma(s,t)$ is a family of curves defining the variation of γ. From Γ we obtain a piecewise smooth vector field $V(t)$ along $\gamma(t)$, called the *variation vector field*, defined as

$$V(t) = \Gamma_s(0,t) := \frac{\partial \Gamma(0,t)}{\partial s}.$$

The variation of $\gamma : [a,b] \to M$ is called *proper* if $V(a) = V(b) = 0$.

(b) The *Symmetry Lemma* on page 154. It says that, restricted to the smooth pieces, the two mixed derivatives of Γ in s and t coincide:

$$\frac{D\Gamma_s}{\partial t} = \frac{D\Gamma_t}{\partial s}.$$

This is essentially Clairaut's theorem from Calculus, which asserts the equality of mixed partial derivatives irrespective of order, although here one covariant derivative is involved. The essential point (in addition to Clairaut's theorem) is the symmetry (torsion-freeness) of the connection.

(c) The *First variation formula*, Theorem 6.3. The essence of the argument is given in the next problem about the Euler-Lagrange equations.

(d) Theorem 6.4 and Corollary 6.5: every length minimizing curve is a geodesic; a unit-speed curve is a critical point of the length functional if and only if it is a geodesic. (Not always minimizing; past the injectivity radius there may be more than one geodesic joining a pair of points. Think of S^n and geodesics having length greater than π.)

(e) Theorem 6.15: Riemannian geodesics are locally length minimizing. A critical ingredient is Gauss's lemma, which is Theorem 6.9.

(f) Theorem 6.19 (Hopf-Rinow): A connected Riemannian manifold is metrically complete if and only if it is geodesically complete.

⋄

5. **The Euler-Lagrange equations.** In this problem we will look at the key variational argument that goes into showing that geodesics are critical points of a length-related functional. But instead of the length functional

itself we consider the so-called *energy* functional, whose critical curves are also geodesics. We will, in fact, be a bit more general and consider a Lagrangean functional that contains a potential function. I'll call these systems *Newtonian-Riemannian.* Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold and $U : M \to \mathbb{R}$ a smooth function, which we call the *potential.* We define the *Lagrangean* $L : TM \to \mathbb{R}$ as the smooth function defined by

$$L(q, v) = \frac{1}{2}\|v\|_q^2 - U(q).$$

The term $\frac{1}{2}\|v\|_q^2$ may be called the *kinetic energy* of the mechanical state indicated by (q, v). M itself describes the space of *configurations* of a mechanical system and TM is the state space. In mechanical systems, the coefficients of the inner product whose norm appears in the kinetic energy contains the mass distribution of the mechanical system. Thus the term $\frac{1}{2}\|v\|_q^2$ has mass terms implicitly (recall the elementary equation $\frac{1}{2}m|v|^2$ for the kinetic energy of a point particle with mass m.)

We now define the *action functional* on curves $\gamma : [a, b] \to M$ joining points $q_1 = \gamma(a)$ and $q_2 = \gamma(b)$:

$$S[\gamma] = \int_a^b L\big(\gamma(t), \gamma'(t)\big)\, dt.$$

The point of this exercise is to find the equation of curves which are critical points of the action functional. We will show that such curves satisfy Newton's equation:

$$\frac{D\gamma'}{dt} = -\text{grad}\, U, \tag{1}$$

where on the left-hand side we have the covariant derivative for the Levi-Civita connection ∇. Recall that the gradient of a function U on M is the unique vector field X such that $\langle X, \cdot\rangle = dU$.

We follow the same ideas used in the textbook for the length functional. Let $\Gamma(s, t)$ be a proper variation of the critical curve $\gamma(t) = \Gamma(0, t)$, where $a \le t \le b$ and $q_1 = \gamma(a) = \Gamma(s, a)$ and $q_2 = \gamma(b) = \Gamma(s, b)$ for all $s \in (-\epsilon, \epsilon)$. In this problem, we assume γ and Γ are smooth instead of piecewise smooth. The variation vector field is defined by

$$V(t) = \Gamma_s(0, t).$$

Note that $V(a) = 0$ and $V(b) = 0$. Now write

$$S[\Gamma(s, \cdot)] = \int_a^b L(\Gamma(s, t), \Gamma_t(s, t))\, dt = \int_a^b \left[\frac{1}{2}\|\Gamma_t(s, t)\|^2 - U(\Gamma(s, t))\right] dt.$$

(a) Using an integration by parts along the way, and the Symmetry Lemma, show that

$$\left.\frac{d}{ds}\right|_{s=0} S[\Gamma(s, \cdot)] = -\int_a^b \left\langle \frac{D\gamma'(t)}{dt} + \text{grad}\, U, V(t) \right\rangle dt.$$

When γ is a critical point of S, and since $V(t)$ is arbitrary, we arrive at the conclusion that γ is a critical point of the action functional if and only if Equation (1) holds. Naturally, if $U = 0$, then critical curves are geodesics.

(b) Now suppose that the Lagrangian function is general and do the same calculation again, but in local coordinates. That is, suppose $\gamma(t) = (x_1(t), \dots, x_n(t))$, $\gamma'(t) = (\dot{x}_1(t), \dots, \dot{x}_n(t))$. Show that the same integration by parts arguments gives the following Euler-Lagrange equations for critical curves:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0, \quad i = 1, \dots, n.$$

(c) Write down the Euler-Lagrange equations for the special case

$$L(x,\dot{x}) = \frac{1}{2}\sum_{i,j} g_{ij}(x)\dot{x}_i\dot{x}_j.$$

Here the $g_{ij}(x)$ are the coefficients of the metric tensor of a Riemannian metric. More specifically, show that those equations are

$$\sum_{i,j}\left(\frac{\partial g_{ki}}{\partial x_j} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x_k}\right)\dot{x}_i\dot{x}_j + \sum_i g_{ki}\ddot{x}_i = 0.$$

(d) Denoting by g^{-1} the inverse of the coefficients matrix $g = (g_{ij})$, it follows from the previous item that

$$\ddot{x}_i + \sum_{k,\ell,j} g^{-1}_{ik}\left(\frac{\partial g_{k\ell}}{\partial x_j} - \frac{1}{2}\frac{\partial g_{\ell j}}{\partial x_k}\right)\dot{x}_\ell\dot{x}_j = 0.$$

Check that this same system of equations can be written in the equivalent form

$$\ddot{x}_i + \frac{1}{2}\sum_{k,\ell,j} g^{-1}_{ik}\left(\frac{\partial g_{k\ell}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_\ell} - \frac{\partial g_{\ell j}}{\partial x_k}\right)\dot{x}_\ell\dot{x}_j = 0.$$

Compare this system with the equations for geodesics in local coordinates (Lee's text: Equations 4.16, page103, and Equation 5.7, page 123.)

6. **Geodesics in conformal metrics.** (You may ignore this problem if it does not seem interesting to you.) Let M be a Riemannian manifold without boundary with metric $\langle\cdot,\cdot\rangle$. You may take M to be an open subset in Euclidean space with the dot-product metric, but the result of this exercise holds in general. Let η be a smooth bounded positive function on M, to be referred to as the *refractive index*. Let E be a constant such that $0 < \eta^2 < E$ and define the function $U := E - \frac{1}{2}\eta^2$. Thus U is another positive smooth function which we call the *potential*. Now define the *optical path length metric*, or simply the *optical metric*

$$\langle\cdot,\cdot\rangle' := \eta^2\langle\cdot,\cdot\rangle.$$

We denote by D and D' the Levi-Civita connections associated to $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ and the respective norms by $\|\cdot\|$ and $\|\cdot\|'$.

According to Fermat's principle, light rays are geodesics of the optical length metric. Let $\gamma(\tau)$ be such a geodesic, where τ indicates arclength parameter for the optical metric. Arclength for the starting metric $\langle\cdot,\cdot\rangle$ will be denoted by s. Let us consider the time change

$$t(\tau) := \int_0^\tau \eta(\gamma(u))^{-2}\,du \tag{2}$$

and write $x(t) := \gamma(\tau(t))$. We refer to t simply as the (mechanical) *time parameter* or*time*. Let us further define the a priori time-dependent quantity

$$m := \left(\frac{\eta(x(t))}{\|\dot{x}(t)\|}\right)^2. \tag{3}$$

Here $\dot{x}$ indicates derivative with respect to t.

(a) Prove the following result, known as Maupertuis's principle.

Proposition 1 (Maupertuis)**.** *Let $\gamma(\tau)$ be a smooth path in M parametrized by arclength relative to the optical metric, and $x(t) = \gamma(\tau(t))$ the same path, now parametrized by t, as defined in Equation (2). Then $\gamma(\tau)$*

is a geodesic with respect to the optical metric if and only if m is constant in time and Newton's second law equation holds:

$$m\frac{D\dot{x}}{dt} = -\text{grad}\, U.$$

(b) Show that if $x(t)$ satisfies Newton's equation with mass parameter m then

$$E := U(x(t)) + \frac{m}{2}\|\dot{x}\|^2,$$

the *total energy* (potential plus kinetic energies, respectively), is a constant of motion.

(c) We record here the identity relating the refractive index and potential function:

$$\eta = \sqrt{2(E-U)}. \tag{4}$$

We could as well have considered geodesics in the optical metric expressed relative to the arclength parameter s associated to the original metric on M. Denoting this path by $x(s)$, show that Newton's equation is equivalent to the so-called *eikonal equation*:

$$\frac{D}{ds}\left(\eta\frac{dx}{ds}\right) = \text{grad}\,\eta. \tag{5}$$

Another fact concerning this general theory that I mention in passing has to do with families of solutions. From wave optics one is naturally led to consider the *eikonal function* S, whose level sets define *wave fronts*. The eikonal function is obtained as a solution to the equation

$$\text{grad}\, S = \eta X, \tag{6}$$

where X is a unit vector field (in the original metric of M, so $\|X\| = 1$) whose integral curves are solutions to the eikonal equation. Note that S satisfies

$$\|\text{grad}\, S\|^2 = \eta^2,$$

which is also called the *eikonal equation*. Light rays are integral curves of the vector field X. The full significance of the function S is only apparent in connection with electromagnetism; from Maxwell's equations one obtains S in terms of the electric and magnetic fields and the so-called Poynting vector.

7. **Killing vector fields. Problems 5-22 and 6-24, pages 150 and 190 of Lee's text.** Let (M, g) be a Riemannian manifold. A vector field $X \in \mathfrak{X}(M)$ is called a *Killing vector field* if the Lie derivative of the metric tensor with respect to X is 0: $\mathscr{L}_X g = 0$. By Proposition B.10 (appendix B of Lee's text) this is equivalent to the condition that the metric tensor g be invariant under the flow of X: $\Phi_t^* g = g$.

   (a) Show that X is a Killing vector field if and only if the covariant 2-tensor $(\nabla X)^\flat$ (a $(0,2)$-tensor field) is anti-symmetric. Note: given $Y, Z \in \mathfrak{X}(M)$, you may take it as a definition that

   $$(\nabla X)^\flat(Y,Z) := \langle \nabla_Y X, Z\rangle.$$

   (b) Prove that a Killing vector field that is normal to a geodesic at one point is normal everywhere along the geodesic.

   (c) Prove that if a Killing vector field vanishes at a point p, then it is tangent to geodesic spheres centered at p. (The concept of *geodesic sphere* is defined on page 158.)

(d) Prove that a Killing vector field on an odd-dimensional manifold cannot have an isolated zero. (You may take for granted the *hairy ball theorem*: There is no nonvanishing continuous tangent vector field on even-dimensional n-spheres.)