Homework set 5 - due 10/04/20

Math 5047 – Renato Feres

This assignment collects a wide range of facts related to formal properties of the Riemannian curvature tensor, spread over several chapters of Lee's text. Make sure to read and think about all the problems, not only the ones I ask you to turn in.

Turn in problems 1(a,b), 2(b) (first Bianchi identity), 4(g), 5(c).

1. **Ricci identities, Lee's text, Proposition 7.14, page 205.** Let ∇ be a torsion-free connection on TM, where the manifold M is not necessarily Riemannian. We know that ∇ extends to a connection on the general tensor bundle by requiring that the product rule holds for the pairing of vectors and covectors and for the tensor product. It thus make sense to define, for a general tensor field τ and $X, Y \in \mathfrak{X}(M)$,

$$R(X,Y)\tau = \nabla_X \nabla_Y \tau - \nabla_Y \nabla_X \tau - \nabla_{[X,Y]}\tau.$$

   Show the following properties of $R(X,Y)$:

   (a) For all $f \in C^\infty(M)$,
$$R(X,Y)f = 0.$$

   (b) For tensor fields τ and η,
$$R(X,Y)(\tau \otimes \eta) = (R(X,Y)\tau) \otimes \eta + \tau \otimes R(X,Y)\eta.$$

   (c) For $\theta \in \Omega^1(M)$ and $Z \in \mathfrak{X}(M)$,
$$(R(X,Y)\theta)(Z) = -\theta(R(X,Y)Z).$$

   (d) For a (k,ℓ)-tensor field τ, covector fields $\theta_1, \dots, \theta_k \in \Omega^1(M)$ and vector fields $X_1, \dots, X_\ell \in \mathfrak{X}(M)$,

$$\begin{aligned} 0 = &(R(X,Y)\tau)(\theta_1, \dots, \theta_k, X_1, \dots, X_\ell) \\ &+ \sum_{i=1}^{k} \tau(\theta_1, \dots, R(X,Y)\theta_i, \dots, \theta_\ell) + \sum_{i=1}^{\ell} \tau(\theta_1, \dots, \theta_k, X_1, \dots, R(X,Y)X_i, \dots, X_\ell). \end{aligned}$$

   (e) If M is Riemannian and ∇ is the Levi-Civita connection we have

$$\langle R(X,Y)W, Z\rangle = -\langle R(X,Y)Z, W\rangle.$$

   Note that (e) is the second property of the next problem (2(b)).

2. **Symmetries of the curvature tensor.** (Lee's text, Proposition 7.12, page 202.) Read Proposition 7.12 and the proof of the symmetries of the curvature tensor of a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$:

   (a) $\langle R(W,X)Y, Z\rangle = -\langle R(X,W)Y, Z\rangle$;

(b) $\langle R(W,X)Y,Z\rangle = -\langle R(W,X)Z,Y\rangle$;

(c) $\langle R(W,X)Y,Z\rangle = \langle R(Y,Z)W,X\rangle$;

(d) $\langle R(W,X)Y,Z\rangle + \langle R(X,Y)W,Z\rangle + \langle R(Y,W)X,Z\rangle = 0$.

The last item is known as the *first Bianchi identity*.

(a) Use these properties to show:

$$\sum_{\sigma\in S_3} \operatorname{sgn}(\sigma) R\left(X_{\sigma(1)}, X_{\sigma(2)}\right) X_{\sigma(3)} = 0,$$

where S_3 is the symmetric group and $\operatorname{sgn}(\sigma)$ is the sign of the permutation $\sigma \in S_3$.

(b) Prove the first Bianchi identity (d).

(c) Show that property (c) is a consequence of the others.

3. **The second Bianchi identity.** (Lee's text, Proposition 7.13, page 204.) Read the proof of the *second Bianchi identity*, Proposition 7.13. The identity is more clearly stated as follows: if ∇ is a torsion-free connection on TM (not necessarily metric), then

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0.$$

Try to give a direct proof of this identity. (This is a bit tedious but more illuminating, I think, than the proof given in Lee's text. That proof uses normal coordinates, which we haven't yet seen.) Recall that

$$(\nabla_X R)(Y,Z) = \nabla_X R(Y,Z) - R(\nabla_X Y, Z) - R(Y, \nabla_X Z).$$

(Convince yourself of this last expression. Recall how ∇, introduced as a covariant derivative on vector fields, gives rise to a covariant derivative on general tensor fields by imposing the requirement that it satisfies the product rule for the tensor product and for the pairing of vector and covector.)

4. **Shape operator, second fundamental form, and Gauss curvature.** Let M be an embedded hypersurface in $\mathbb{R}^{n+1}$. (So $\dim(M) = n$.) We assume M is oriented. This can be shown to imply the existence of a global unit length vector field N on M such that $N(p)$ is perpendicular to T_pM at each $p \in M$. The *shape operator* of M at $p \in M$ is the linear map $S_p : T_pM \to T_pM$ defined by

$$S_p u = -D_u N$$

for $u \in T_pM$. Here D is the Euclidean connection in $\mathbb{R}^{n+1}$. It should be intuitively clear that S_p, by measuring changes in the normal vector N along M, should somehow encode the bending, or curving, of M in Euclidean space. This is made more precise here.

(a) Show that $D_u N \in T_pM$ for all $u \in T_pM$. So S_p is indeed a linear map from T_pM to itself.

(b) Show that $\langle S_p X, Y\rangle = \langle D_X Y, N\rangle$ for any $X, Y \in \mathfrak{X}(M)$. (In particular, the expression on the right depends on the values of X and Y at p.)

(c) Show that S_p is symmetric: for all $u, v \in T_pM$,

$$\langle S_p u, v\rangle_p = \langle u, S_p v\rangle_p.$$

Here the inner product is the ordinary dot product in Euclidean space restricted to T_pM. Therefore, S_p is self-adjoint. In particular, the spectral theorem for self-adjoint (finite dimensional) operators applies, and implies that T_pM has an orthonormal basis of eigenvectors of S_p.

(d) Let us define the *principal directions* of the hypersurface M at p as the unit eigenvectors of S_p. The *principal curvatures* are the eigenvalues of S_p. Describe the shape operator of the sphere

$$S^n(r) = \left\{x \in \mathbb{R}^{n+1} : |x| = r\right\}$$

of radius r and determine its principal curvatures.

(e) Find the principal directions and principal curvatures at any point on the cylinder of radius r in $\mathbb{R}^3$:

$$\left\{x = (x_1, x_2, x_3) : x_1^2 + x_2^2 = r^2\right\}.$$

(f) Let M be a surface in $\mathbb{R}^3$ with the Riemannian metric induced on TM from the dot product on $\mathbb{R}^3$. Let R be the curvature tensor on M. Let us suppose that E_1, E_2 constitute a local orthonormal frame on a neighborhood of $p \in M$ such that, at each q in that neighborhood, $E_1(q), E_2(q)$ are principal directions with principal curvatures $\lambda_1(p), \lambda_2(q)$. Show that

$$\langle R(E_1, E_2)E_2, E_1\rangle = \lambda_1\lambda_2.$$

The product of the principal curvatures at a point p of a surface in $\mathbb{R}^3$ is the *Gauss curvature* of the surface at that point. Note the remarkable fact that, given the Riemannian metric on M (induced from $\mathbb{R}^3$), the quantity $\langle R(E_1, E_2)E_2, E_1\rangle$ is obtained intrinsically, without further reference to the ambient space, whereas λ_i has to do with the bending of the surface in the Euclidean space. (Note that λ_i changes sign if we choose the normal vector field to be $-N$, but the product $\lambda_1\lambda_2$ is unaffected.) In particular, Gauss's curvature is invariant under local isometry. This fact is the statement of Gauss's Theorema Egregium.

(g) Let e_1, e_2 be an orthonormal basis for a plane σ in T_pM where M is a Riemannian manifold. Show that $K(\sigma) := \left\langle R_p(e_1, e_2)e_2, e_1\right\rangle$ does not depend on the choice of orthonormal basis. It is called the *sectional curvature* of M at p associated to the plane σ. Find the value of $K(\sigma)$ at any point $p \in S^n(r)$ and for any plane $\sigma \subseteq T_pS^n(r)$. We say that the sphere of radius r has *constant sectional curvature*. What is the relation between the sectional curvature of the sphere and its principal curvatures?

(h) The *second fundamental form* of a hypersurface M in Euclidean space with unit normal vector field N is the bilinear form on each T_pM defined by

$$\mathbb{I}_p(u, v) = \langle S_p u, v\rangle N_p.$$

Note that the second fundamental form is symmetric since the shape operator is self-adjoint. More generally, suppose that M is a submanifold of Euclidean space of arbitrary codimension. Define

$$\mathbb{I}_p(u, v) = (D_u v)^\perp.$$

In words, $\mathbb{I}_p(u, v)$ is the normal component of the Euclidean covariant derivative of any local extension of v in the direction of u. It is not difficult to show that $\mathbb{I}_p(u, v)$ is tensorial in u and v and does not depend on the local extension. Check that

$$\langle R(u, v)w, z\rangle = \langle \mathbb{I}(u, z), \mathbb{I}(v, w)\rangle - \langle \mathbb{I}(u, w), \mathbb{I}(v, z)\rangle.$$

5. **Curvature and differential forms.** (This exercise is related to Problems 4-14 page 113 and 7-5 page 222, of Lee's text.) Let $E_1, \ldots, E_n$ be a local orthonormal frame on a Riemannian manifold. Let $\theta_1, \ldots, \theta_n$ be the coframe of

smooth 1-forms characterized by the equations $\theta_i(E_j) = \delta_{ij}$. Note that θ_i is the one form given by $X \mapsto \langle E_i, X\rangle$. If ∇ is the Levi-Civita connection, define the connection 1-forms by

$$\omega_{ij}(X) = \langle E_i, \nabla_X E_j\rangle = \theta_i(\nabla_X E_j).$$

(a) Show that $\omega_{ij} = -\omega_{ji}$. Since $\omega^\intercal = -\omega$, we say that ω is a Lie algebra-valued one-form, for the Lie algebra $\mathfrak{o}(n)$ of the orthogonal group $O(n)$.

(b) Prove Cartan's first structure equation:

$$d\theta_i + \sum_j \omega_{ij} \wedge \theta_j = 0.$$

(c) Define the curvature 2-forms Ω_{ij} as $\Omega_{ij}(X,Y) = \left\langle E_i, R(X,Y)E_j\right\rangle$, where R is the Riemann curvature tensor:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Show that

$$\Omega_{ij} = d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj}.$$

6. **Conformally Euclidean Riemannian manifolds.** Let M be a non-empty open subset of $\mathbb{R}^n$ and let $\eta : M \to (0,\infty)$ be a smooth positive function. We define a Riemannian metric on M by

$$\langle u, v\rangle_p = \frac{1}{[\eta(p)]^2}\, u \cdot v$$

where $u, v \in T_pM$ and $u \cdot v$ denotes the ordinary dot product of vectors in $\mathbb{R}^n$, where we identify T_pM with $\mathbb{R}^n$. We define an orthonormal frame on M consisting of the vector fields $E_i = \eta\frac{\partial}{\partial x_i}$, $i = 1, \dots, n$, where the partial derivatives are with respect to the standard coordinates in $\mathbb{R}^n$. Introduce the dual frame of 1-forms $\theta_1, \dots, \theta_n$, the connection 1-forms ω_{ij}, and the curvature 2-forms Ω_{ij} as in the previous problem. We use the notation $\eta_i = \frac{\partial\eta}{\partial x_i}$ and $\eta_{ij} = \frac{\partial^2\eta}{\partial x_i \partial x_j}$.

(a) Show that $[E_i, E_j] = \eta_i E_j - \eta_j E_i$.

(b) Show that $d\theta_i = \theta_i \wedge d\log\eta = \sum_j \eta_j \theta_i \wedge \theta_j$.

(c) Show that $\omega_{ij} = \eta_i\theta_j - \eta_j\theta_i$. (There is more than one way to find the connection forms. One convenient way is to obtain the values of $\omega_{ij}(E_k)$ from the expression for the Levi-Civita connection

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\{X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle\}$$

for any vector fields X, Y, Z on M.)

(d) Check that the first structure equation $d\theta_i + \sum_j \omega_{ij} \wedge \theta_j = 0$ holds for this example.

(e) Using the second structure equation $\Omega_{ij} = d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj}$, show that

$$\Omega_{ij} = \eta \sum_k \left(\eta_{jk}\theta_i - \eta_{ik}\theta_j\right) \wedge \theta_k - \left(\sum_k \eta_k^2\right) \theta_i \wedge \theta_j.$$

(f) Suppose $M = \left\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\right\}$ and that $\eta(x_1, \dots, x_n) = x_n$. Show that $\Omega_{ij} = -\theta_i \wedge \theta_j$.

(g) Using the previous item, show that M (the upper-half space, or *real hyperbolic space*) has constant sectional curvature equal to -1.

7. **Ricci and scalar curvatures.** (See Chapter 7, Page 207, Lee's text.) Let M be a Riemannian manifold and R the Riemann curvature tensor (obtained from the Levi-Civita connection). The *Ricci curvature tensor* is defined, at each $p \in M$, as the trace of the linear map $w \in T_pM \mapsto R(w,u)v \in T_pM$:

$$\mathrm{Ric}_p(u,v) := \mathrm{tr}\,(w \mapsto R(w,u)v)\,,$$

for all $u, v \in T_pM$. This defines a smooth tensor field Ric. If $E_1, \dots, E_n$ is a local frame on M (not necessarily orthonormal) and $\theta_1, \dots, \theta_n$ is the dual frame of 1-forms (so that $\theta_i(E_j) = \delta_{ij}$) then

$$\mathrm{Ric}(X,Y) = \sum_k \theta_k(R(E_k,X)Y).$$

If the local frame is orthonormal, then

$$\mathrm{Ric}(X,Y) = \sum_k \langle E_k, R(E_k,X)Y\rangle\,.$$

It is clear (given the trace definition) that the tensor defined by the last two expressions does not depend on the choice of local frame.

The *scalar curvature* is a smooth function S on M defined as follows. Let $E_1, \dots, E_n$ be a local orthonormal frame. Then

$$S = \sum_k \mathrm{Ric}\,(E_k, E_k)\,.$$

(a) Show that the Ricci curvature tensor is symmetric: $\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X)$.

(b) Show that the scalar curvature does not depend on the choice of orthonormal frame.

(c) Show that the Ricci tensor for the real hyperbolic space (upper-half space of the previous problem) is

$$\mathrm{Ric}_p(u,v) = -(n-1)\langle u,v\rangle_p$$

where n is the dimension of M.

(d) Show that the scalar curvature of the real hyperbolic space is $S = -n(n-1)$.

(e) You may ignore this item if you don't find it sufficiently interesting. (We won't need it later in the course.) Let R be the Riemann curvature tensor of a Riemannian manifold M and ∇ the Levi-Civita connection. We define the *covariant divergence* of the Ricci tensor at a point $p \in M$ as

$$\mathrm{div}(\mathrm{Ric})(u) = \sum_i (\nabla_{e_i}\mathrm{Ric})(e_i,u),$$

where $e_1, \dots, e_n$ is an orthonormal basis of T_pM and $u \in T_pM$. (The covariant divergence can be defined for more general tensor fields in a similar way.) Show that

$$\mathrm{div}(\mathrm{Ric}) = \frac{1}{2}dS.$$

For more information on this equation and its connection with general relativity theory, see the remarks on 211 of Lee's text. This formula is stated in Proposition 7.18, page 209. Take a look at the proof by Penrose tensor diagrams given in my notes on (current) page 346.

8. **Geometry of the orthogonal group.** Let us recall a fact from the homework assignment 1, problem 6. The orthogonal group $O(n)$ is the matrix Lie group consisting of orthogonal $n \times n$ matrices: $g^\top g = I$. The Lie algebra

$\mathfrak{o}(n)$ may be described in two isomorphic ways: as the space of left-invariant vector fields on $O(n)$

$$X_A(x) = \sum_{i,j} (xA)_{ij} \frac{\partial}{\partial x_{ij}}$$

where A is an antisymmetric $n \times n$ matrix: $A^\intercal = -A$. The Lie algebra operation is the Lie bracket of vector fields. This Lie algebra is isomorphic to the space of antisymmetric $n \times n$ matrices with the commutator bracket: $[A, B] = AB - BA$. It was proved in problem 6 of homework 1 that these two Lie algebras are isomorphic under the correspondence $A \mapsto X_A$:

$$[X_A, X_B] = X_{[A,B]}.$$

Note that at the identity element e (equal to the identity matrix I)

$$X_A(e) = A = \sum_{i,j} A_{ij} \frac{\partial}{\partial x_{ij}}.$$

Also recall that the flow of X_A is given by the exponential map:

$$\Phi_t(g) = ge^{tA}.$$

In particular, if $L_g : h \mapsto gh$ and $R_g : h \mapsto hg$ denote left- and right-multiplication in $O(n)$,

$$d(L_g)_e X_A(e) = \left.\frac{d}{dt}\right|_{t=0} ge^{tA} = gA = \sum_{i,j} (gA)_{ij} \frac{\partial}{\partial x_{ij}}$$

and

$$d(R_g)_e X_A(e) = \left.\frac{d}{dt}\right|_{t=0} e^{tA}g = Ag = \sum_{i,j} (Ag)_{ij} \frac{\partial}{\partial x_{ij}}.$$

Note that the push-forward $(R_g)_* X_A$ is also a left-invariant vector field since left and right multiplication commute. In fact, convince yourself that

$$R_{g*} X_A = X_{g^{-1}Ag}.$$

Let us define a Riemannian metric on $O(n)$ using the trace as follows: on left-invariant vector fields X_A, X_B,

$$\langle X_A, X_B \rangle_g := \frac{1}{2} \operatorname{tr}\left(AB^\intercal\right).$$

It is not difficult to check that this expression, extended by bilinearity to general vector fields, is indeed a Riemannian metric on $O(n)$.

(a) It is immediate from this definition that the Riemannian metric is left-invariant. Show that it is also right-invariant. We say that this Riemannian metric is *bi-invariant*.

(b) Show that the Levi-Civita connection ∇ for this metric satisfies the property

$$\nabla_{X_A} X_B = \frac{1}{2} X_{[A,B]}.$$

(Use the equation (5.5) in the proof of Theorem 5.10, page 122, of Lee's text.)

(c) Show that

$$R(X_A, X_B)X_C = -\frac{1}{4} X_{[[A,B],C]}.$$

(d) Show that the sectional curvature $K(\sigma)$ of the plane spanned by X_A, X_B is

$$K = \frac{1}{8}\frac{\operatorname{tr}([A,B][A,B]^\top)}{Q(A,B)},$$

where

$$Q(A,B) = \|X_A\|^2\|X_B\|^2 - \langle X_A, X_B\rangle^2 = \frac{1}{16}\left(\operatorname{tr}\left(AA^\top\right)\operatorname{tr}\left(BB^\top\right) - 2\left(\operatorname{tr}\left(AB^\top\right)\right)^2\right).$$

Note: If A, B are orthonormal, we may write (identifying X_A and A)

$$K = \frac{1}{4}\|[A,B]\|^2.$$

(e) The Lie algebra of $O(3)$ is generated by the matrices

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is not difficult to check that these elements form an orthonormal basis of $\mathfrak{o}(3)$ and

$$[L_1, L_2] = L_3, \quad [L_2, L_3] = L_1, \quad [L_3, L_1] = L_2.$$

Let $u = \sum_i a_i L_i$ and $v = \sum_i b_i L_i$ be an orthonormal basis of a plane σ in $T_e O(n)$. Show that

$$K(\sigma) = \frac{1}{4}|a \times b|^2 = \frac{1}{4}.$$

where $|a \times b|$ is the Euclidean norm of the cross-product from Calc. III.

Note: the connected component of $O(3)$, namely, $SO(3)$, is double covered by the sphere S^3. (Topologically, $SO(3)$ is homeomorphic to projective space $\mathbb{R}P^3$.) So it should not be surprising that the sectional curvature is constant.

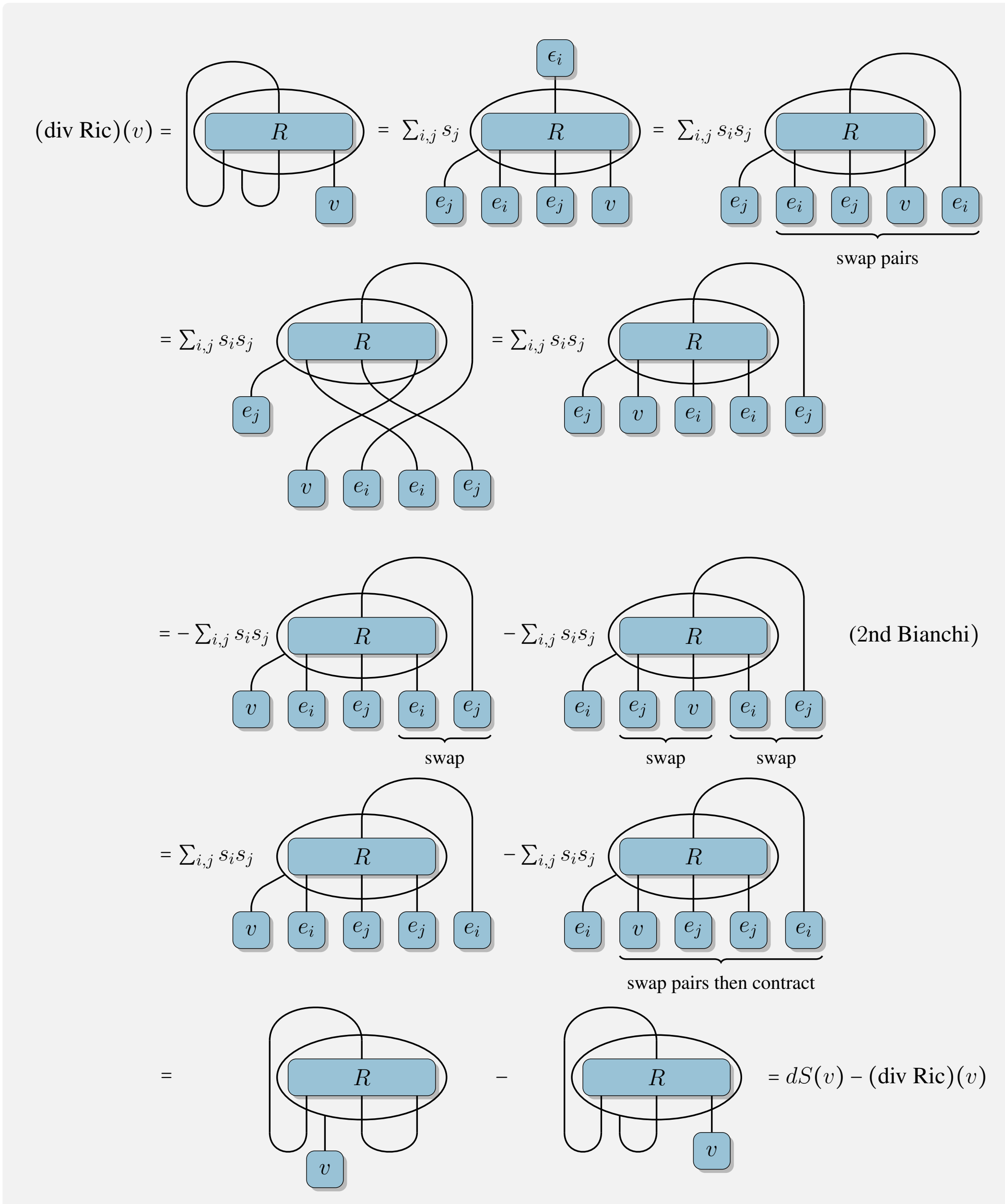

Figure 1: Proof of Equation $dS = 2\,\text{div}(\text{Ric})$. If you are interested in figuring out what is going on in this figure, take a look at my notes on tensor calculus. For a Riemannian metric tensor, the s_i are all 1. (This proof allows for semi-Riemannian metrics, for which $s_i = \langle e_i, e_i \rangle = \pm 1$.) The very first step in the proof is now show: the trace operation involved in the definition of the Ricci tensor commutes with the divergence operator. (The lasso around R at the very beginning goes through the wire loop defining the trace.)