Homework set 4 - due 10/27/20

Math 5047 – Renato Feres

Turn in problems 1, 2, 3, 4.

1. **Example of a geodesically incomplete connection.** We consider an affine connection ∇ on the (tangent bundle of the) 1-dimensional torus, $\mathbb{T}$. Equivalently, we may work in $\mathbb{R}$ and assume that the single Christoffel symbol $\Gamma(x)$, defined by
$$\nabla_{\frac{d}{dx}} \frac{d}{dx} = \Gamma(x)\frac{d}{dx},$$
is periodic: $\Gamma(x+1) = \Gamma(x)$. Let $x(t)$ be a geodesic relative to ∇ with initial conditions $x(0) = 0$ $\dot{x}(0) = 1$, where $\dot{x}(t)$ is the derivative in t.

   (a) Show that $x(t)$ satisfies the differential equation
$$\ddot{x} + \Gamma(x)\dot{x}^2 = 0.$$

   (b) Suppose that $\Gamma(x) = a$ is a constant. Find an equation for $x(t)$.

   (c) Assuming $\Gamma(x) = a$, for which values of a is ∇ geodesically complete? (That is, for which a are geodesics defined for all time t?)

   (d) Now suppose the torus $\mathbb{T}$ is equipped with a Riemannian metric and that ∇ (for a not necessarily constant $\Gamma(x)$) is a metric connection. Let us write
$$\left\langle \frac{d}{dx}, \frac{d}{dx} \right\rangle = g(x)$$
for the single metric coefficient, where $g(x)$ is periodic: $g(x+1) = g(x)$. The metric connection condition means
$$\frac{d}{dx}g(x) = \frac{d}{dx}\left\langle \frac{d}{dx}, \frac{d}{dx} \right\rangle = 2\left\langle \nabla_{\frac{d}{dx}} \frac{d}{dx}, \frac{d}{dx} \right\rangle = 2\Gamma(x)g(x).$$
Show that
$$\int_0^1 \Gamma(x)\,dx = 0.$$
Also show that $x(t)$ satisfies the differential equation
$$\dot{x}(t)\sqrt{g(x(t))} = \text{const.} = \sqrt{g(0)}. \tag{1}$$
In this case, as $g(x)$ is periodic, hence bounded and bounded away from 0, it is not difficult to show that $x(t)$ is defined for all t. Hence, in the metric case, ∇ is geodesically complete. Note: for any value of $x \in \mathbb{R}$, let $t(x)$ be given by the following equation, obtained by integrating Equation (1):
$$t(x) = \int_0^x \sqrt{\frac{g(s)}{g(0)}}\,ds.$$

It is clear from this equation that t can take any real value.

2. **Riemannian metric on projective space $\mathbb{R}P^n$.**

   (a) Prove that the antipodal mapping $A: S^n \to S^n$ given by $A(p) = -p$ is an isometry of $S^n = \{p \in \mathbb{R}^{n+1} : |p| = 1\}$.

   (b) Use this fact to introduce a Riemannian metric on the real projective space $P^n(\mathbb{R})$ such that the natural projection $\pi: S^n \to P^n(\mathbb{R})$ is a local isometry.

3. **Isometric immersion of the flat torus in Euclidean space.** Obtain an isometric immersion of the flat torus $\mathbb{T}^n$ into $\mathbb{R}^{2n}$. Suggestion: start with $n = 1$ and recall that $\mathbb{T}^n$ is the Cartesian product of n copies of $\mathbb{T}^1$.

4. **Conformally Euclidean metrics.** Let $\eta(\mathbf{x})$ be a positive function defined on an open subset $\mathcal{U}$ in $\mathbb{R}^n$ and define the Riemannian metric on $\mathcal{U}$

$$\langle u, v \rangle_x := \eta^2(x) u \cdot v.$$

   We say that the metric is conformally Euclidean.

   (a) Express the Riemannian volume of a compact subset $A \subseteq \mathcal{U}$ as an integral over A.

   (b) Express the length of a differentiable curve $c(t) \in \mathcal{U}$, $t \in [a,b]$, as an ordinary integral over the interval $[a,b]$.

5. **Geodesics of a surface of revolution.** Denote by (u, v) the Cartesian coordinates of $\mathbb{R}^2$. Define the function $\varphi: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\varphi(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$$

   where $U = (u_0, u_1) \times (v_0, v_1)$, f and g are differentiable functions with $f'(v)^2 + g'(v)^2 \neq 0$ and $f(v) \neq 0$.

   (a) Show that φ is an immersion. (The image $\varphi(U)$ is the surface generated by the rotation of the curve $(f(v), g(v))$ around the axis $0z$ and is called a *surface of revolution* S. The image by φ of the curves $u =$ constant and $v =$ constant are called *meridians* and *parallels*, respectively, of S.)

   (b) Show that the induced metric in the coordinates (u, v) is given by

$$g_{11} = f^2, \quad g_{12} = 0, \quad g_{22} = f'^2 + g'^2.$$

   (c) Show that local equations of a geodesic γ are

$$\frac{d^2u}{dt^2} + \frac{2ff'}{f^2}\frac{du}{dt}\frac{dv}{dt} = 0$$

$$\frac{d^2v}{dt^2} - \frac{ff'}{f'^2 + g'^2}\left(\frac{du}{dt}\right)^2 + \frac{f'f'' + g'g''}{f'^2 + g'^2}\left(\frac{dv}{dt}\right)^2 = 0.$$

   (d) Obtain the following geometric meaning of the equations above: the second equation is, except for meridians and parallels, equivalent to the fact that the "energy" $|\gamma'(t)|^2$ of a geodesic is constant along γ; the first equation signifies that if $\beta(t)$ is the oriented angle, $\beta(t) < \pi$, of γ with a parallel P intersecting γ at $\gamma(t)$, then

$$r\cos\beta = \text{ constant},$$

   where r is the radius of the parallel P. (The equation above is called *Clairaut's relation.*)

(e) (You don't have to do this part. If you are curious, see do Carmo's *Differential Geometry of Curves and Surfaces*, Dover Publications, 2016, page 262.) Use Clairaut's relation to show that a geodesic of the paraboloid

$$\left(f(v) = v, g(v) = v^2, v > 0, -\epsilon < u < 2\pi + \epsilon\right)$$

which is not a meridian, intersects itself an infinite number of times. (See Figure 6 on page 79.)