Homework set 3 - due 09/20/24

Math 5047

Turn in problems 3, 4, 5, 6.

1. **Read all of Chapter 4 of Lee's text.**

2. **Read Chapter 5 of Lee's text up to page 26 (end of section 'Connections on Abstract Riemannian Manifolds').**

3. **Parallel transport.** Let M be a Riemannian manifold. Consider the mapping

$$P = P_{c,t_0,t} : T_{c(t_0)}M \to T_{c(t)}M$$

defined by: $P_{c,t_0,t}(v)$, $v \in T_{c(t_0)}M$, is the vector obtained by parallel transporting the vector v along the curve c. Show that P is an isometry and that, if M is oriented, P preserves the orientation.

4. **Recovering the connection from parallel transport.** Let X and Y be differentiable vector fields on a Riemannian manifold M. Let $p \in M$ and let $c : I \to M$ be an integral curve of X through p, i.e. $c(t_0) = p$ and $\frac{dc}{dt} = X(c(t))$. Prove that the Riemannian connection of M satisfies

$$(\nabla_X Y)(p) = \frac{d}{dt}(P_{c,t_0,t})^{-1}(Y(c(t)))\bigg|_{t=t_0},$$

where $P_{c,t_0,t} : T_{c(t_0)}M \to T_{c(t)}M$ is the parallel transport along c, from t_0 to t. (This shows how the connection can be recovered from the concept of parallelism.)

5. **Levi-Civita connection of a submanifold.** Let $f : M^n \to \overline{M}^{n+k}$ be an immersion of a differentiable manifold M into a Riemannian manifold $\overline{M}$. Assume that M has the Riemannian metric induced by f. That is, the Riemannian inner product on T_pM is the restriction to T_pM of the Riemannian inner product on $T_p\overline{M}$. Let $p \in M$ and let $U \subseteq M$ be a neighborhood of p such that $f(U) \subseteq \overline{M}$ is a submanifold of $\overline{M}$. Further, suppose that X, Y are differentiable vector fields on $f(U)$ which extend to differentiable vector fields $\overline{X}, \overline{Y}$ on an open set of $\overline{M}$. Define

$$(\nabla_X Y)(p) = \text{ tangential component of } \left(\overline{\nabla}_{\overline{X}}\overline{Y}\right)(p),$$

where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{M}$. Prove that ∇ is the Levi-Civita connection of M.

6. **Covariant derivative of vector field over a constant curve.** Let M be a Riemannian manifold and let p be a point of M. Consider a constant curve $f : I \to M$ given by $f(t) = p$, for all $t \in I$. Let V be a vector field along f (that is, V is a differentiable mapping of I into T_pM). Show that $\frac{DV}{dt} = \frac{dV}{dt}$, that is to say, the covariant derivative coincides with the usual derivative of $V : I \to T_pM$.

7. **Horizontal and vertical subbundles of TTM.** This extended discussion contains a very useful characterization of connections on a vector bundle N. We will be a bit sketchy. See, for example, *Differential Geometric Structures* by Walter P. Poor for details omitted here.

We begin by considering $N = TM$, the tangent bundle of a smooth manifold M, and $\pi : N \to M$ the base-point projection. It will be convenient to indicate a point $(p, v) \in N$ simply by v, with $\pi(v) = p$. At each $v \in N$ define the *vertical* subspace $\mathcal{V}_v \subseteq T_v N$ to be the kernel of $d\pi_v : T_v N \to T_p M$. The disjoint union $\mathcal{V} = \coprod_{v \in TM} \mathcal{V}_v$ with the natural base-point projection $\mathcal{V} \to N$, which maps $\xi \in \mathcal{V}_v$ to v, is easily shown to be a smooth vector subbundle of TN. Notice that $\mathcal{V}_v$ is the tangent space at v of the fiber $\pi^{-1}(p)$, where $p = \pi(v)$.

In general, there is no canonical way to select a complementary subbundle to $\mathcal{V}$, that is, a vector subbundle $\mathcal{H}$ of TN such that $TN = \mathcal{V} \oplus \mathcal{H}$ (direct sum of vector bundles). A choice of such *horizontal* subbundle is (essentially) a choice of connection on TM, as you will show.

(a) Show that each vertical subspace $\mathcal{V}_v \subseteq T_v N$ is canonically isomorphic to $T_p M$, $p = \pi(v)$. Specifically, check that the map

$$\mathcal{I}_v : w \in T_p M \mapsto \left.\frac{d}{dt}\right|_{t=0} (v + tw) \in \mathcal{V}_v$$

is an isomorphism of vector spaces.

(b) Let ∇ be an affine connection on TM. For each $v \in N$, $p = \pi(v) \in M$, and X a smooth vector field on an open subset of M containing p such that $X(p) = v$, consider the linear map

$$\mathcal{K}_v : w \in T_p M \mapsto dX_p w - \mathcal{I}_v \nabla_w X \in T_v N.$$

Check that: (1) $\mathcal{K}_v$ only depends on the value of X at p; and (2) it satisfies

$$d\pi_v \circ \mathcal{K}_v = \mathrm{id}_{T_p M}.$$

The image $\mathcal{H}_p$ of $T_p M$ under this map is called the *horizontal* subspace of $T_v N$ associated to the connection ∇. It follows from the second part of this item that the $\mathcal{H}_p$ have dimension $n = \dim M$ and

$$d\pi_v : \mathcal{H}_v \to T_p M$$

is a linear isomorphism. It can be shown (no need to prove it here) that $v \mapsto \mathcal{H}_v$ is a smooth vector subbundle of TN called the *horizontal* subbundle.

(c) For each $v \in N$, $p = \pi(v)$, and $w \in T_p M$, let $t \mapsto c(t)$ be a smooth curve in M representing w (i.e., such that $c(0) = p$, $c'(0) = w$). Let $V(t)$ be the parallel transport of v along $c(t)$ in M. Then $t \mapsto V(t)$ is a smooth curve in N such that $V(0) = v$ and $V'(0) \in T_v N$. Check that $V'(0) \in \mathcal{H}_v$ and

$$d\pi_v V'(0) = c'(0) = w.$$

(d) Check that $T_v N = \mathcal{V}_v \oplus \mathcal{H}_v$, the direct sum of the vertical and horizontal subspaces. (Since both $\mathcal{V}_v$ and $\mathcal{H}_v$ have dimension $\dim(M)$, which is half of the dimension of N, it suffices to check that $\mathcal{V}_v \cap \mathcal{H}_v = \{0\}$.) This direct sum decomposition allows us to define the linear map

$$K_v : T_v N \to T_p M \cong \mathcal{I}_p T_p M = \mathcal{V}_v$$

called the *connection map*. Thus the kernel of K_v is $\mathcal{H}_v$ and $K_v \circ \mathcal{I}_p$ is the identity map on $T_p M$.

(e) For each $a \in \mathbb{R}$, let $\mu_a : N \to N$ be defined by $\mu_a(v) = av$. Show that $d(\mu_a)_v : T_v N \to T_{av} N$ maps $\mathcal{H}_v$ to $\mathcal{H}_{av}$. We say that the horizontal subbundle is *homogeneous*. (The vertical subbundle is also homogeneous.)

(f) (This remark won't be needed later, but you may find it interesting if you saw modules in an algebra course.)

A short exact sequence of modules over a fixed ring

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$$

is called *split exact* if it is isomorphic to the exact sequence:

$$0 \to A \xrightarrow{i} A \oplus C \xrightarrow{p} C \to 0$$

where i is inclusion and p is projection. What we have obtained above is that an affine connection on TM gives rise to a splitting of the short exact sequence of modules over $C^\infty(M)$:

$$\mathfrak{X}(M) \xrightarrow{\mathscr{I}} \Gamma(TN) \xrightarrow{\pi_*} \mathfrak{X}(M),$$

so that $\Gamma(TN) \cong \Gamma(\mathscr{V}) \oplus \Gamma(\mathscr{H})$, a direct sum of modules over $C^\infty(M)$.

(g) Let $\mathscr{H}$ be any complementary vector subbundle of $\mathscr{V}$ in TN that satisfies the homogeneity property of the above item (7e). Let

$$K_v : T_v N = \mathscr{V}_v \oplus \mathscr{H}_v \to \mathscr{V}_v \cong T_p M$$

be the projection to the vertical subspace, which is canonically isomorphic to $T_p M$. Show that

$$\nabla_w X := K_v dX_p w,$$

for all $X \in \mathfrak{X}(M)$, defines a connection on TM. Check that parallel translation with respect to this connection has the following description: Let $t \mapsto c(t)$ be a smooth curve on M and $v \in T_{c(0)}M$. Then c has a unique lift $\overline{c}(t)$, a curve in N such that $\pi \circ \overline{c}(t) = c(t)$, $\overline{c}(0) = v$, and $\overline{c}'(t) \in \mathscr{H}_{\overline{c}(t)}$. Note that $\overline{c}(t)$ can be viewed as a vector field along $c(t)$ in M.

(h) Convince yourself that all that we have done above works just as well if we replace TM with a general vector bundle $\pi : N \to M$. That is, a connection on N is equivalent to a splitting of TN into a direct sum $\mathscr{V} \oplus \mathscr{H}$ of vertical and horizontal bundles, where for each $e \in N$, $\mathscr{V}_e$ is canonically isomorphic to the fiber N_p, $p = \pi(e)$, and $\mathscr{H}_e$ is homogeneous and isomorphic to $T_p M$. Parallel transport in M amounts to the horizontal lifting of curves in M.

8. **Parallel transport on S^2.** Let $S^2 \subseteq \mathbb{R}^3$ be the unit sphere, c an arbitrary parallel of latitude on S^2 and v_0 a tangent vector to S^2 at a point of c. Convince yourself that the following claim holds. Consider the cone tangent to S^2 that intersects S^2 at c. Then the parallel transport of v_0 along c is the same whether it is performed on the sphere or on the cone. Furthermore, parallel translation on the cone is obtained by cutting it along a ray and unrolling it flat on $\mathbb{R}^2$. Then parallel translation on the cone is then simply vector space translation in $\mathbb{R}^2$.

9. **Review of exterior calculus (or calculus of differential forms).** The following are all definitions.

(a) **Interior product**. Let X be a vector field on the smooth manifold M. Given a k-form θ on M, we define the $(k-1)$-form $i_X\theta$ to be 0 if $k = 0$ and

$$(i_X\theta)(Y_1, \ldots, Y_{k-1}) = \theta(X, Y_1, \ldots, Y_{k-1})$$

if $k \geq 1$, where $Y_1, \ldots, Y_{k-1}$ are smooth vector fields.

(b) **Lie derivative of differential forms.** Let X be a smooth vector field on a smooth manifold M. Recall that to X one can associate its (local) flow Φ_t such that for any given $p \in M$, $\Phi_t(p)$ is the integral curve of X with initial condition $\Phi_0(p) = 0$, defined for some open interval in t containing 0. In other words, $\gamma(t) = \Phi_t(p)$ satisfies the initial value problem

$$\gamma'(t) = X_{\gamma(t)}, \ \ \gamma(0) = p.$$

From the general theory of differential equations we know that there is a unique solution (over a maximal interval for t) and that Φ_t defines a local flow of diffeomorphisms: $\Phi_{t+s} = \Phi_t \circ \Phi_s$ whenever the composition makes sense. In addition, Φ_0 is the identity diffeomorphism. (Under certain conditions, e.g., when the manifold is compact, we know that Φ_t is a diffeomorphism of M for all $t \in \mathbb{R}$. In such cases we say that Φ_t defines a flow on M.) If θ is a differential form on M, it makes sense (for t sufficiently close to 0) to define the pullback $\Phi_t^*\theta$. The *Lie derivative* of θ with respect to X at $p \in M$ is defined by

$$(\mathscr{L}_X\theta)_p := \lim_{t\to 0} \frac{(\Phi_t^*\theta)_p - \theta_p}{t} = \frac{d}{dt}(\Phi_t^*\theta)_p\Big|_{t=0}.$$

A similar definition applies to vector fields if we define $\Phi_t^* Y := (\Phi_{-t})_* Y$, where Y is a smooth vector field on M. (With a little thought, it is not difficult to figure out how to define the Lie derivative for a general tensor field. We may return to this later.)

10. **Interior multiplication as a signed derivation.** Show that i_X satisfies the following signed product rule:

$$i_X(\theta \wedge \eta) = i_X\theta \wedge \eta + (-1)^k \theta \wedge i_X\eta$$

where θ is a k-form and η is an arbitrary differential form. (This is a pointwise operation; there are no derivatives actually involved. Note: You can find the proof of this fact in John Lee's "Introduction to Smooth Manifolds," second edition, Lemma 14.13, page 358. You can get a pdf of this text from the Olin library.)

11. **Properties of the Lie derivative**

(a) Show that if f is a smooth function on M (a 0-form) then $\mathscr{L}_X f = Xf$.

(b) Show that $\mathscr{L}_X$ commutes with the exterior derivative: $\mathscr{L}_X d = d\mathscr{L}_X$.

(c) Show that if θ_1, θ_2 are arbitrary smooth forms,

$$\mathscr{L}_X(\theta_1 \wedge \theta_2) = \mathscr{L}_X\theta_1 \wedge \theta_2 + \theta_1 \wedge \mathscr{L}_X\theta_2$$

(d) Show that

$$\mathscr{L}_X Y = [X, Y].$$

I suggest the following approach: first recall that if Y is a vector field on a manifold M and $F : M \to N$ is a smooth map into another manifold N, then the derivative map $F_{*,p} : T_pM \to T_{F(p)}N$ satisfies

$$(F_{*,p}Y_p)f = Y_p(f \circ F)$$

where f is any smooth function on N. (Convince yourself of this basic fact!) Let now F be a diffeomorphism. Then the push-forward of Y under F is defined as the vector field F_*Y on N given at any $q = F(p) \in N$ by

$$(F_*Y)_q = F_{*,p}Y_p.$$

Now suppose that f and Y are vector fields on N. It follows from the above remarks and the definition of pull-back of functions ($F^* f = f \circ F$) that

$$(Yf)(F(p)) = (F_*^{-1} Y)_p (F^* f)(p).$$

Now let $F = \Phi_t$ (the local flow of X) and take derivatives in t at $t = 0$.

(e) If θ is a differential k-form and $X, Y_1, \dots, Y_k$ are smooth vector fields,

$$X\theta(Y_1, \dots, Y_k) = (\mathscr{L}_X \theta)(Y_1, \dots, Y_k) + \theta(\mathscr{L}_X Y_1, Y_2, \dots, Y_k) + \cdots + \theta(Y_1, \dots, Y_{k-1}, \mathscr{L}_X Y_k).$$

As a suggestion, you may begin by observing that if $F : M \to N$ is a diffeomorphism between smooth manifolds, θ a k-form on N and $Y_1, \dots, Y_k$ vector fields on N, then

$$(\theta(Y_1, \dots, Y_k)) \circ F = (F^* \theta)\left(\left(F^{-1}\right)_* Y_1, \dots, \left(F^{-1}\right)_* Y_k\right).$$

Now let $N = M$, set $F = \Phi_t$ the local flow of X, and take the derivative in t on both sides of the equation.

(f) Show that for any smooth form θ and smooth vector field X,

$$\mathscr{L}_X \theta = (d i_X + i_X d)\theta.$$

Here are some potentially useful remarks: From the general properties of the exterior derivative (See the beginning of A.5, page 299 of Tu's text), if θ is a k-form and ω is a general smooth form, then

$$d(\theta \wedge \omega) = d\theta \wedge \omega + (-1)^k \theta \wedge d\omega.$$

Using the previous problem concerning the interior product, show that the operation $\mathscr{M}_X = d i_X + i_X d$ satisfies

$$\mathscr{M}_X(\theta \wedge \omega) = \mathscr{M}_X \theta \wedge \omega + \theta \wedge \mathscr{M}_X \omega.$$

We need to show that $\mathscr{M}_X = \mathscr{L}_X$. From the previous remark, it suffices to show this equality of operations on smooth functions and 1-forms of the type $f\,dg$. Now, both $\mathscr{L}_X$ and $\mathscr{M}_X$ commute with d so, in fact, it suffices to check that the operations are the same on functions.

(g) Show that if θ is a differential 1-form and X_1, X_2 are smooth vector fields, then

$$d\theta(X_1, X_2) = X_1 \theta(X_2) - X_2 \theta(X_1) - \theta([X_1, X_2]).$$

Suggestion: express this identity in terms of the exterior derivative, the Lie derivative, and the interior product.

(h) Show that if θ is a smooth 2-form and X_1, X_2, X_3 are smooth vector fields, then

$$d\theta(X_1, X_2, X_3) = X_1 \theta(X_2, X_3) - X_2 \theta(X_1, X_3) + X_3 \theta(X_1, X_2) - \theta([X_1, X_2], X_3) + \theta([X_1, X_3], X_2) - \theta([X_2, X_3], X_1).$$

Suggestion: follow the same pattern of proof as the previous problem.