Homework set 2 - due 09/13/24

Math 5047

Turn in problems 2(b,c), 3, 4(a,b), 5(a,b).

1. **Reading.** Read Chapter 4 of Lee's text, pages 84-103.

2. **Problem 4-1, page 111, Lee.** Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold and $Y \in \mathfrak{X}(M)$. For every point $p \in M$ and vector $v \in T_pM$, define $\nabla_v Y$ by
$$\nabla_v Y = \Pi_p D_v \tilde{Y},$$
where $\tilde{Y}$ is a smooth vector field defined on an open subset of $\mathbb{R}^n$ containing p and $\Pi_p : T_p\mathbb{R}^n \to T_pM$ is the orthogonal projection (relative to the Euclidean dot-product in $\mathbb{R}^n$).

   (a) Show that $\nabla_v Y$ does not depend on the choice of extension $\tilde{Y}$ of Y. [Lee offers a hint, which I find confusing, so I won't write it here.]

   (b) It is already shown in Example 4.9 that ∇ is an affine connection on TM. Now show that it is torsion free:
$$\nabla_X Y - \nabla_Y X = [X, Y],$$
for $X, Y \in \mathfrak{X}(M)$. (Local extensions of X, Y exist by Exercise A.23.)

   (c) Show that ∇ is compatible with the induced Riemannian metric on M. That is, show that for $X, Y \in \mathfrak{X}(M)$ and $v \in T_pM$,
$$v(X \cdot Y) = (\nabla_v X) \cdot Y + X \cdot \nabla_v Y$$
where $\cdot$ is the ordinary dot-product on $\mathbb{R}^n$. This makes sense since tangent vectors to M may be regarded as vectors in $\mathbb{R}^n$.

   (d) Show that ∇ is invariant under rigid motions of $\mathbb{R}$, in the following sense: if F is an element of the Euclidean group (the group of transformations of $\mathbb{R}^n$ generated by rotations and translations), then
$$dF_p \nabla_v Y = \nabla_{dF_p v} F_* Y.$$
Here F maps M to $F(M)$ and we use the same notation ∇ for the connection on TM and on $TF(M)$.

3. **Problem 4-6, page 112, Lee.** Let M be a smooth manifold and let ∇ be a connection in TM. Define a map $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by
$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

   (a) Show that T is a $(1,2)$-tensor field, called the *torsion tensor* of ∇. The connection is called *symmetric* or *torsion free* if $T = 0$. For example, the connection of Problem 2 is symmetric.

(b) Let $\frac{\partial}{\partial x_i}$ be coordinate vector fields on a local coordinate chart of M. Define the *Christoffel symbols* Γ_{ij}^k as the functions given by

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

Show that ∇ is symmetric if and only if on every coordinate chart, $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all i, j, k.

4. **Problem 4-11, page 113, Lee, expanded.** Suppose G is a connected Lie group.

(a) Show that there is a unique connection ∇ on TG with the property that every left-invariant vector field is parallel. (Note: a vector field X is parallel relative to an affine connection ∇ if $\nabla X = 0$.)

(b) Show that ∇ is torsion free if and only if G is abelian. (For this assignment, take for granted that if the Lie algebra is abelian then the connected Lie group is abelian. We'll discuss this further in class.)

(c) Define τ to be the $(1,2)$-tensor field on G such that on $u, v \in T_g G$,

$$\tau_g(u, v) = [X_u, X_v]_g,$$

where X_u is the unique left-invariant vector field on G equal to u at g. (Convince yourself that τ is, indeed, a tensor field and that $\tau(X, Y) = [X, Y]$ on left-invariant vector fields.) Now define a connection on G as follows:

$$\overline{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\tau(X, Y).$$

Thus if X, Y are left-invariant, $\overline{\nabla}_X Y = \frac{1}{2}[X, Y]$. Show that $\overline{\nabla}$ is still a connection on TM and it is torsion free. (Even if G is non-abelian.)

(d) Show that both ∇ and $\overline{\nabla}$ are invariant under left-translations. That is, show the following: let $F = L_g : G \to G$ be the diffeomorphism of G such that $F(h) = gh$ for all $g, h \in G$. This is the *left translation map*. Then

$$F_* \nabla_X Y = \nabla_{F_* X} F_* Y.$$

(e) Show that both ∇ and $\overline{\nabla}$ are also invariant under right translations: if $F = R_g : G \to G$, $F(h) = hg$, then

$$F_* \nabla_X Y = \nabla_{F_* X} F_* Y.$$

For this, note that if X is a left-invariant vector field, then $(R_g)_* X$ is also left-invariant. (This is because R_g and L_h commute for all $g, h \in G$, something easy to check.)

5. **Problem 4-14, page 113, Lee.** Let M be a smooth n-manifold and ∇ a connection in TM, let $(E_1, \ldots, E_n)$ be a local frame on some open subset $U \subseteq M$, and let $(\epsilon_1, \ldots, \epsilon_n)$ be the dual coframe.

(a) Show that there is a uniquely determined $n \times n$ matrix of smooth 1-forms (ω_{ij}) on U, called the *connection 1-forms* for this frame, such that

$$\nabla_X E_i = \sum_j \omega_{ij}(X) E_j$$

for all $X \in \mathfrak{X}(M)$.

(b) *Cartan's first structure equation.* Prove that these forms satisfy the following equation, due to Élie Cartan:

$$d\epsilon_j = \sum_{i=1}^n \epsilon_i \wedge \omega_{ij} + \tau_j,$$

where $\tau_1, \ldots, \tau_n \in \Omega^2(M)$ are the *torsion* 2*-forms*, defined in terms of the torsion tensor and the local frame by:

$$T(X,Y) = \sum_{i=1}^{n} \tau_i(X,Y) E_i.$$