Homework set 1 - due 09/06/24

Math 5047

- For most of my problems, answers will consist of pointing to a theorem or theorems and performing some calculation. I mainly want to see that you can do the calculations and are aware of the pertinent theorems. Try to be clear and succinct in your writing. (The grader will be grateful!)
- Begin to work on these assignments early and feel free to ask questions about them in class. I can't be sure of what you'd have seen in the first two semesters of Geometry/Topology, or even whether I'm using familiar notation. Always feel free to bring up for class discussion whatever seems confusing or possibly related to material I may be assuming you know that was not covered in Geometry/Topology I and II.
- Assignments will typically be due Friday at 11:59PM. They will handled via Gradescope. (I'm not familiar with it myself! I hope to figure it out before this assignment's due date.) Whenever possible, I plan to take a few minutes from class every Wednesday to discuss the assignment due Friday. Make an effort to start working on it early to make the Wednesday discussion most helpful to you.
- Homework sets will contain a variable number of problems, but only 4 will actually be graded. (I don't want to overwhelm the grader with work.) Solutions to all of them will be available after the due date.
- The final exam will be based on material taken from homework assignment questions.

Problems

Turn in your solutions to Exercises 2, 3(a), 6, 7.

1. **Reading.** Do a careful browsing of the appendices A, B, C of Lee's text.
2. **Spherical coordinates.** Using the standard notation (x,y,z) for the Cartesian coordinates on $\mathbb{R}^3$, let V be the complement in $\mathbb{R}^3$ of the half-plane defined by $y=0$ and $x\geq 0$: Let U be the open box $(0,\infty)\times(0,2\pi)\times(0,\pi)$, with coordinates (ρ,θ,φ). The spherical coordinates on $\mathbb{R}^3$ are given by the following f:

$$x = f_1(\rho,\theta,\varphi) = \rho\sin\varphi\cos\theta$$
$$y = f_2(\rho,\theta,\varphi) = \rho\sin\varphi\sin\theta$$
$$z = f_3(\rho,\theta,\varphi) = \rho\cos\varphi.$$

   Find an expression for $f_*\frac{\partial}{\partial\rho}$, $f_*\frac{\partial}{\partial\theta}$ and $f_*\frac{\partial}{\partial\phi}$ in the coordinates (x,y,z).

3. **The groups $SL(n,\mathbb{R})$, $O(n)$, and $SO(n)$.** The sets

$$SL(n,\mathbb{R}) = \{A\in M(n,\mathbb{R}) : \det A = 1\},\quad O(n) = \{A\in M(n,\mathbb{R}) : A^\intercal A = I\}$$

are subgroups of the general linear group $GL(n,\mathbb{R})$ of invertible real matrices of size n-by-n. They are called, respectively, the *special linear* and the *orthogonal* groups. The intersection $SO(n) = SL(n,\mathbb{R}) \cap O(n)$ is called the *special orthogonal* group.

(a) Show that $SL(n,\mathbb{R})$ is a Lie group. For this, first show that the differential of the determinant function $\det : M(n,\mathbb{R}) \to \mathbb{R}$ at any non-singular (i.e., invertible) matrix A is surjective.

(b) Show that $O(n) = \{A \in M(n,\mathbb{R}) : A^\intercal A = I\}$ is a Lie group. For this, first show that the map $f(A) = A^\intercal A$ from $M(n,\mathbb{R})$ into the space of symmetric, positive, n-by-n matrices is a submersion at each $A \in O(n)$.

(c) Explain that $SO(n) = O(n) \cap SL(n,\mathbb{R})$ is a Lie group.

4. **The sphere S^n.** The n-dimensional sphere is the set

$$S^n = \{q \in \mathbb{R}^{n+1} : \|q\| = 1\}.$$

(a) Show that S^n is a smooth submanifold of $\mathbb{R}^{n+1}$. Do this by showing that for all non-zero q the differential at q of the map $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $f(q) = \|q\|^2$ is surjective.

(b) Explain why the quotient $SO(n+1)/SO(n)$ admits a smooth manifold structure with respect to which the projection map $SO(n+1) \to SO(n+1)/SO(n)$ is smooth.

(c) The special orthogonal group $SO(n+1)$ acts smoothly on the sphere S^n according to the action map $(A, x) \mapsto Ax$. Let $\mathcal{N} = (0, \ldots, 0, 1)^\intercal$ denote the north pole. We may regard $SO(n)$ as the subgroup

$$SO(n) = \{A \in SO(n+1) : A\mathcal{N} = \mathcal{N}\}.$$

Now define a map $\varphi : SO(n+1)/SO(n) \to S^n$ by $\varphi : gSO(n) \mapsto g\mathcal{N}$. Show that this map is a diffeomorphism.

5. **Linear vector fields.** Consider vector fields on $\mathbb{R}^n$ of the form $X_A(p) = -Ap$ where $A = (a_{ij})$ is an $n \times n$ real matrix. Here we regard p as a column vector. These are called *linear vector fields*. In terms of the basis $\{\partial_1, \ldots, \partial_n\}$, where $\partial_i = \frac{\partial}{\partial x_i}$, X can be written as follows:

$$X_A(p) = -\sum_{ij} a_{ij} x_j(p) \partial_i.$$

Given two $n \times n$ matrices A and B, show that

$$[X_A, X_B] = X_{[A,B]}$$

where $[A, B] = AB - BA$ is the matrix commutator.

6. **The Lie algebra of $O(n)$.** Recall that $O(n)$ is the orthogonal group in dimension n, which consists of n-by-n real matrices R satisfying $R^\intercal = R^{-1}$. We already know that $O(n)$ is a Lie group relative to matrix multiplication and inverse. Let

$$\mathfrak{o}(n) = \{A \in M(n,\mathbb{R}) : A^\intercal = -A\}.$$

For each $A \in \mathfrak{o}(n)$ define the vector field on $M(n,\mathbb{R})$ expressed in the coordinates $x = (x_{ij})$ by

$$X_A(x) = \sum_{ij} (xA)_{ij} \frac{\partial}{\partial x_{ij}}.$$

(a) Check that $\mathfrak{o}(n)$, with the matrix commutator $[A, B] = AB - BA$, is a Lie algebra.

(b) Show that $[X_A, X_B] = X_{[A,B]}$ for all $A, B \in \mathfrak{o}(n)$.

(c) Show that the flow associated to X_A is given by $\Phi_t(g) = ge^{tA}$ for all $g \in O(n)$.

(d) Show that $X_A(g) \in T_g O(n)$ for all $g \in O(n)$.

(e) Show that X_A is a left-invariant vector field on $O(n)$. For this, show that

$$(dL_g)_h X_A(h) = X_A(gh).$$

(f) Conclude that the Lie algebra of left-invariant vector fields on $O(n)$ is isomorphic to $\mathfrak{o}(n)$ with the commutator bracket of matrices.

7. **The tensorial property.** Let $M = \mathbb{R}^n$. Let Z be a smooth vector field on M. At every $p \in M$ let

$$\Pi_p : T_p M \to T_p M$$

denote the orthogonal projection to $Z_p^\perp = \{v \in T_p M : Z_p \cdot v = 0\}$. It is not difficult to check that Π is a smooth tensor field on M of type $(1,1)$. Now define $\tau : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$,

$$\tau(X,Y) = Z \cdot [\Pi X, \Pi Y].$$

Here $\cdot$ is the ordinary dot product in $\mathbb{R}^n$. In other words, at each point p, $\tau(X,Y)$ computes the orthogonal projection to the subspace of $T_p M$ spanned by Z_p of the Lie bracket of the vector fields ΠX and ΠY (both everywhere orthogonal to Z). Show that τ is a smooth tensor field of type $(0,2)$. We also say that τ is *tensorial* in the arguments X, Y.

8. **Transformation of multilinear maps under a diffeomorphism.** Show that if $f : M \to N$ is a diffeomorphism, θ a smooth tensor field of type $(0,k)$ on M, and $X_1, \dots, X_k \in \mathfrak{X}(M)$, then $f_*(\theta(X_1,\dots,X_k)) = (f_*\theta)(f_* X_1, \dots, f_* X_k)$. (Recall that $f_* h = h \circ f^{-1}$ for $h \in C^\infty(M)$.)

9. **Flows of commuting vector fields.** Let $X_1, \dots, X_n$ be everywhere linearly independent and commuting smooth vector fields defined on an open set $U \subset M$ of a smooth, n-dimensional manifold M. (There is no loss of generality in assuming that $M = \mathbb{R}^n$.) Let $\Phi_{t_i}^{X_i}$ be the local flow of X_i. We assume that $V \subset U$ is an open subset and $I = (-a, a)$ is a small enough interval such that $\Phi_{t_i}^{X_i}(p)$ is defined for all $p \in V$ and all $(t_1, \dots, t_n) \in I^n$. For a fixed $p_0 \in V$ define

$$\varphi(t_1,\dots,t_n) := \left(\Phi_{t_1}^{X_1} \circ \cdots \circ \Phi_{t_n}^{X_n}\right)(p_0).$$

Show that φ is a diffeomorphism from I^n onto its image in V such that $\varphi^* X_i = \frac{\partial}{\partial t_i}$. Therefore, the X_i are the coordinate vector fields of the coordinate neighborhood of p_0 defined by φ.