Homework set 7 - due 10/29/21

Math 5031

These problems involve the parts of Kerr's notes up to, including, homomorphisms of rings (III.E). You should do all of them, but will be asked to turn in numbers 1, 2, 3, 4 and 5.

1. Let R be a commutative ring and N denote the set of nilpotent elements of R. Show:
   (a) N is an ideal;
   (b) R/N contains no non-zero nilpotent elements.
2. Let $\eta : R \to R'$ be a homomorphism of rings.
   (a) Given a unit $u \in R^*$, show that $\eta(u)$ is a unit in R'.
   (b) Suppose that η is surjective. Does this imply that η maps R^* onto the group of units of R'? (Suggestion: consider the rings $\mathbb{Z}$ and $\mathbb{Z}_p$.)
3. Let R be a commutative ring of prime characteristic p.
   (a) Show that $\phi : R \to R$ defined by $a \mapsto a^p$ is a homomorphism (also called an *endomorphism* of R).
   (b) Describe the map ϕ when $R = \mathbb{Z}_p$.
   (c) Describe the map ϕ when $R = \mathbb{Z}_p[x]$.
   (d) Is ϕ always an automorphism? (An *automorphism* of R is a (ring) isomorphism from R to itself.)
4. Let F be a finite field of characteristic p (a prime). Let $\varphi : \mathbb{Z} \to F$ be the ring homomorphism defined by $\varphi(m) = 1 + \cdots + 1$, where $1 \in F$ is added m times.
   (a) Show that the kernel of φ is $p\mathbb{Z}$ and its image is isomorphic to $\mathbb{Z}_p$.
   (b) Show that $p-1$ divides $|F|-1$. (Hint: consider the group of units of $\mathbb{Z}_p$ in $F^* = F \setminus \{0\}$.)
   (c) If $|F|$ is even, show that F has characteristic 2.
5. If S is a subset of a ring R, then the subring generated by S is defined to be the intersection of all the subrings containing S. If this is R itself then S is called a *generating set* for R.
   (a) Show that the subring generated by S is the subset of R consisting of all the finite sums of finite products of elements in S.
   (b) Show that if η_1 and η_2 are homomorphisms from R to R' which agree on the elements of a generating set $S \subset R$, then $\eta_1 = \eta_2$.
6. In Homework Assignment 6, Exercise 6, you showed that $R = \mathbb{Z}\left[\frac{1+\sqrt{-d}}{2}\right]$ is an Euclidean domain for $d = 3$. The same argument used there shows that R is an Euclidean domain for $d = 3, 7, 11$. For this problem, let $R = \mathbb{Z}[\alpha]$, where $\alpha = \frac{1+\sqrt{-19}}{2}$. Show that R is not Euclidean. (That is, show that R admits no Euclidean function δ.)

   I suggest the following steps:

(a) Explain: $\mathbb{Z}[\alpha] \cong \frac{\mathbb{Z}[x]}{(x^2-x+5)}$. (See Example III.E.6 of Kerr's notes.)

(b) Determine the group of units R^*. (This was obtained in HW 6.)

(c) Suppose that an Euclidean function $\delta : R\setminus\{0\} \to \mathbb{N}$ exists. Let s be an element of $R\setminus(R^* \cup \{0\})$ that minimizes δ on this set. Show that $R/(s)$ is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. (A commutative ring with no more than 3 elements must contain $0, 1, -1$, with the possibility that $1 = -1$.)

(d) Obtain a contradiction by showing that $\frac{\mathbb{Z}[x]}{(x^2-x+5)}$ does not admit a quotient isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

7. Show that the following identities involving ideals in $\mathbb{Z}[\sqrt{10}]$ hold:

(a) $\left(1+\sqrt{10}\right) = \left(3, 1+\sqrt{10}\right)^2$

(b) $\left(-1+\sqrt{10}\right) = \left(3, -1+\sqrt{10}\right)^2$

(c) $(3) = \left(3, 1+\sqrt{10}\right)\left(3, -1+\sqrt{10}\right)$