Homework 5

Math 308

Due: 1 March

Guidelines:

- You are strongly encouraged to work together to understand the problems, but what you turn in must be your own work.
- Your submission must be clearly written and stapled. Homework will only be accepted up to the beginning of lecture, or you can drop it off at my office before class.

(1) (6.2.2)

(a) Show that if f and g are functions with second derivatives, then $u = f(x - vt) + g(x + vt)$ is a solution of the wave equation; this is called the d'Alembert solution and is physically interpreted as a superposition of two waves moving in opposite directions.

(b) Show that $u(r,t) = \frac{1}{r}f(r-vt) + \frac{1}{r}g(r+vt)$ is a solution of the wave equation in spherical coordinates. (See, e.g. p. 294 for the gradient in spherical coordinates). Again, this can be interpreted in terms of waves moving in and out from the origin.

Solution. This is a special case of a more general method known as the *Method of Characteristics*, which is a technique for solving hyperbolic equations. This class of equations is a natural generalization of the wave equation. In essence, we can get equations like this by reducing the problem to a one-dimensional setting (represented by the variable $x \pm vt$), at which point we're left with an ODE that can be studied with more elementary methods.

(a) This is an application of the chain rule. Note that

$$\frac{\partial u}{\partial t} = f'(x - vt) \cdot \frac{d(x - vt)}{dt} + g'(x - vt) \cdot \frac{d(x + vt)}{dt} = -vf'(x - vt) + vg'(x - vt).$$

Taking a second derivative gives

$$\frac{\partial^2 u}{\partial t^2} = v^2 \left(f''(x - vt) + g''(x - vt)\right).$$

Proceeding in the same way gives us

$$\frac{\partial^2 u}{\partial x^2} = f''(x - vt) + g''(x - vt)$$

from which the claim follows.

(b) In spherical coordinates, the Laplacian is given (see p. 298) by

$$\nabla^2 F = \frac{1}{r^2}(r^2 F_r)_r + \frac{1}{r^2 \sin\theta}(\sin\theta F_\theta)_\theta + \frac{1}{r^2 \sin^2\theta} F_{\varphi\varphi}.$$

Because of the angular symmetries here, any term that involves a φ or θ derivative vanishes and we're left with

$$\nabla^2 F = \frac{1}{r^2}(r^2 F_r)_r = \frac{1}{r^2}(r^2 F_{rr} + 2rF_r) = F_{rr} + \frac{2}{r}F_r.$$

So now we can compute the Laplacian of u. Let's start with

$$\left(\frac{f(r-vt)+g(r+vt)}{r}\right)_r = -\frac{f(r-vt)+g(r+vt)}{r^2} + \frac{f'(r-vt)+g'(r+vt)}{r}.$$

and

$$\left(\frac{f(r-vt)+g(r+vt)}{r}\right)_{rr} = 2\frac{f(r-vt)+g(r+vt)}{r^3} - 2\frac{f'(r-vt)+g'(r+vt)}{r^2} + \\ + \frac{f''(r-vt)+g''(r+vt)}{r}.$$

Putting it all together,

$$\begin{aligned}\nabla^2 u &= 2\frac{f(r-vt)+g(r+vt)}{r^3} - 2\frac{f'(r-vt)+g'(r+vt)}{r^2} + \frac{f''(r-vt)+g''(r+vt)}{r} + \\ &+ \frac{2}{r}\left(-\frac{f(r-vt)+g(r+vt)}{r^2} + \frac{f'(r-vt)+g'(r+vt)}{r}\right) \\ &= \frac{f''(r-vt)+g''(r+vt)}{r}.\end{aligned}$$

On the other hand, the t-derivative is much easier to compute and we already did it in part (a). It is immediate then that

$$\nabla^2 u = \frac{1}{v^2} u_{tt}$$

and we're done.

(2) (See 6.3.2) A bar of length 10 centimeters is initially at 100°, and starting at time $t = 0$ the ends are held at 0°. Find the temperature distribution at time t. For any time t, determine where the maximum temperature in the bar is. Roughly how long does it take for the temperature to drop to 1°? What about 0.1°?

Solution. We'll apply separation of variables. Collecting our information, we have

$$\begin{aligned} u_{xx} &= u_t \qquad \text{for } 0 < x < 10, t > 0 \\ u(x,0) &= 100 \qquad \text{for } 0 < x < 10 \\ u(0,t) = u(10,t) &= 0 \qquad \text{for } t > 0. \end{aligned}$$

Following our examples in class, we can reduce to

$$\frac{X''}{X} = -\lambda^2 = \frac{T'}{T}$$

which leads us to $X(x) = A\sin\lambda x + B\cos\lambda x$ and $T(t) = Ce^{-\lambda^2 t}$. The boundary condidition $X(0) = 0$ eliminates the cosine terms, while $X(10) = 0$ implies that 10λ is an integer multiple of π; thus the corresponding eigenvalues are $\{0, \pi/10, 2\pi/10, ...\}$. Superimposing, we are looking for a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n\pi/10)^2 t} \sin\left(\frac{n\pi x}{10}\right).$$

In order to find the Fourier coefficients c_n, we set $t = 0$ and use the initial condition:

$$100 = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{10}\right).$$

Multiplying by $\sin(m\pi x/10)$ and integrating from 0 to 10, we get

$$\int_0^{10} 100\sin\left(\frac{m\pi x}{10}\right)\,dx = \sum_{n=1}^{\infty} c_n \int_0^{10} \sin\left(\frac{n\pi x}{10}\right)\sin\left(\frac{m\pi x}{10}\right)\,dx = 5c_m.$$

The final inequality follows from computing the integral, and finding that it is zero for $m \neq n$, and $\frac{1}{2} \cdot 10$ if $m = n$. Therefore,

$$c_m = 20 \int_0^{10} \sin\left(\frac{m\pi x}{10}\right) dx = -\frac{200(\cos(m\pi) - 1)}{m\pi} = \frac{200}{\pi}\left(\frac{1 - \cos(m\pi)}{m}\right)$$

Using the fact that $\cos(m\pi) = (-1)^m$, this is zero for m even; otherwise, $1 - \cos(m\pi) = 2$. Finally, this leads to

$$\boxed{u(x,t) = \frac{400}{\pi} \sum_{\substack{n \geq 1 \\ \text{odd}}} \frac{1}{n} e^{-(n\pi/10)^2 t} \sin\left(\frac{n\pi x}{10}\right).}$$

The highest temperature in the bar should be at the center, because the bar is dissipating heat from either end and the temperature distribution is symmetric. Therefore, the maximum temperature at time t is just

$$u(5,t) = \frac{400}{\pi} \sum_{\substack{n \geq 1 \\ \text{odd}}} \frac{1}{n} e^{-(n\pi/10)^2 t} \sin\left(\frac{n\pi}{2}\right).$$

Since the sine term vanishes when n is even and alternates between ± 1 otherwise, we have that the maximum temperature is

$$u(5,t) = \frac{400}{\pi}\left(e^{-(\pi/10)^2 t} - \frac{1}{3} e^{-(3\pi/10)^2 t} + \frac{1}{5} e^{-(5\pi/10)^2 t} - \frac{1}{7} e^{-(7\pi/10)^2 t} + \cdots\right)$$

This series converges very rapidly because of the growth of the exponents and is very well approximated by its first term. In fact, after $t = 20$, the error involved is less than 3×10^{-6} degrees, due to the alternating series test. Therefore,

$$u(5,t) \approx \frac{400}{\pi} e^{-(\pi/10)^2 t}.$$

The temperature at the midpoint therefore drops to $1°$ after $t \approx 49$, and $0.1°$ after $t \approx 72$. The temperature drops by a factor of 10 roughly every 23 time units.

(3) (See 6.3.8) A bar of length 2 is initially at temperature $0°$. From $t = 0$ on, the $x = 0$ end is held at $0°$ and the $x = 2$ end is held at $100°$. Find the time dependent temperature distribution $u(x,t)$. Then compute $\lim_{t\to\infty} u(x,t)$ for any x; explain what this means physically.

Solution. We again proceed by separation of variables. From our physical reasoning, we know that the steady-state temperature distribution must be

$$\lim_{t\to\infty} u(x,t) = 50x$$

in order to linearly interpolate the boundary data at each side of the bar. (Alternatively, in the steady state case we are left with $u_{xx} = 0$ with $u(0) = 0$ and $u(2) = 100$). Subtract this term to get $w(x,t) = u(x,t) - 50x$; this is still a solution to the heat equation, and our problem is now

$$\begin{aligned} w_{xx} &= w_t && \text{for } 0 < x < 2, t > 0 \\ w(x,0) &= -50x && \text{for } 0 < x < 10 \\ w(0,t) &= 0 && \text{for } t > 0 \\ w(2,t) &= 0 && \text{for } t > 0. \end{aligned}$$

Following the process of the previous problem, we end up with

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n\pi/2)^2 t} \sin\left(\frac{n\pi x}{2}\right).$$

Now using our initial data, we have

$$-50x = w(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right).$$

The Fourier coefficients are therefore given by

$$c_m = \frac{\int_0^2 -50x \sin\left(\frac{m\pi x}{2}\right)\, dx}{\int_0^2 \sin^2\left(\frac{m\pi x}{2}\right)\, dx} = \frac{200(m\pi\cos(m\pi) - \sin(m\pi))/(m^2\pi^2)}{1} = (-1)^m \frac{200}{m\pi}$$

Putting it all together, we have $u = 50x + w$, so

$$u(x,t) = 50x + \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-(n\pi/2)^2 t} \sin\left(\frac{n\pi x}{2}\right).$$

Physically, we have the steady state term $50x$, together with an exponentially decaying transient term that disippates quite quickly.

(4) (6.2.16) Suppose we have a closed region D and a harmonic function u which takes certain values on the boundary of the region. That is, we have a solution to the PDE

$$\begin{cases} \nabla^2 u &= 0 \quad \text{in } D \\ \quad u &= f \quad \text{on } \partial D \end{cases}$$

Show that there is only one such function (that is, the solution with given boundary data is *unique*). To do this, suppose that u_1 and u_2 are two solutions; then $U := u_1 - u_2$ satisfies Laplace's equation with zero boundary data. Use Green's first identity (from a previous homework) with the function U to show that $\nabla U \equiv 0$. Then explain why this is enough.

Solution. Recall that Green's first identity said that

$$\iiint_D (\varphi\nabla^2\psi + \nabla\varphi\cdot\nabla\psi)\, dV = \oiint_{\partial D} (\varphi\nabla\psi)\cdot\vec{n}\, d\sigma.$$

Apply this with $\varphi = \psi = u_1 - u_2$. Note that ψ is harmonic, so $\nabla^2\psi = 0$. Moreover, φ vanishes on the boundary (see the remark in the problem statement!), so the integral on the right is zero. This leads us to the conclusion

$$\iiint_D \nabla U\cdot\nabla U\, dV = 0.$$

Now the integrand here is just $|\nabla U|^2$, which is always nonnegative. A continuous function which is nonnegative but has zero integral is necessarily zero everywhere, so $\nabla U \equiv 0$.

This is enough to prove the claim, because a function with zero gradient must be constant. The boundary values of U are zero, and so this constant must be zero. Unpacking, we have

$$U \equiv 0 \implies u_1 - u_2 \equiv 0 \implies u_1 \equiv u_2$$

as desired.