Homework 4

Math 308

Due: 15 February

Guidelines:

- You are strongly encouraged to work together to understand the problems, but what you turn in must be your own work.
- Your submission must be clearly written and stapled. Homework will only be accepted up to the beginning of lecture, or you can drop it off at my office before class.

(1) (6.11.2) Given $\vec{A} = (x^2 - y^2)\vec{\imath} + 2xy\vec{\jmath}$, verify Stokes' theorem on a rectangle in the xy-plane bounded by the lines $x = 0, y = 0, x = a$, and $y = b$. That is, compute both the surface integral of the curl and the corresponding line integral and verify their equality.

Solution. The line integral $\oint \vec{A} \cdot d\vec{r}$ can be computed in four parts, corresponding to the four parts of the boundary (going counterclockwise from the origin):

$$\int_{\text{bottom}} \vec{A} \cdot d\vec{r} = \int_0^a \langle t^2, 0\rangle \cdot \langle 1, 0\rangle \, dt = \frac{1}{3}a^3$$

$$\int_{\text{right}} \vec{A} \cdot d\vec{r} = \int_0^b \langle a^2 - t^2, 2at\rangle \cdot \langle 0, 1\rangle \, dt = ab^2$$

$$\int_{\text{top}} \vec{A} \cdot d\vec{r} = \int_0^a \langle (a-t)^2 - b^2, 2(a-t)b\rangle \cdot \langle -1, 0\rangle \, dt = \int_0^a b^2 - (a-t)^2 \, dt = ab^2 - \frac{1}{3}a^3$$

$$\int_{\text{left}} \vec{A} \cdot d\vec{r} = \int_0^b \langle -(b-t)^2, 0\rangle \cdot \langle 0, -1\rangle \, dt = 0$$

Adding all four pieces, we have

$$\oint \vec{A} \cdot d\vec{r} = 2ab^2.$$

Next, we compute the surface integral. The outward normal vector here is $\vec{n} = \vec{k}$ (corresponding to the standard positive orientation of the rectangle); thus we only need to compute the z-component $(\nabla \times \vec{A}) \cdot \vec{k}$ rather than the full curl. We find

$$(\nabla \times \vec{A})_z = \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2 - y^2) = 4y.$$

Therefore

$$\iint (\nabla \times \vec{A}) \cdot \vec{n} \, d\sigma = \int_0^a \int_0^b 4y \, dy \, dx = 2ab^2$$

agrees with the previous value.

(2) (6.12.20) Compute

$$\iint \vec{P} \cdot \vec{n} \, d\sigma$$

on the upper half of the sphere of radius 1 centered at $(0, 0, 0)$, where $\vec{P} = \nabla \times \langle 0, x, -z\rangle$.

Solution. The boundary of the surface is the circle $x^2+y^2=1, z=0$, which inherits a counterclockwise orientation from the orientation of the sphere. Hence

$$\begin{aligned}\iint \vec{P}\cdot\vec{n}\,d\sigma &= \oint \langle 0,x,-z\rangle\cdot d\vec{r}\\ &= \int_0^{2\pi} \langle 0,\cos t,0\rangle\cdot\ d\langle\cos t,\sin t,0\rangle\\ &= \int_0^{2\pi} \cos^2 t\,dt\\ &= \pi\end{aligned}$$

To check this, notice that $\vec{P}=\langle 0,0,1\rangle$ and $\vec{P}\cdot\vec{n}=z$. Hence, the integral can be written in spherical coordinates, giving

$$\int_0^{\pi/2}\int_0^{2\pi}(\cos\varphi)\,d\theta\ \sin\varphi\,d\varphi = 2\pi\int_0^{\pi/2}\sin\varphi\cos\varphi\,d\varphi=\pi$$

as before.

(3) (6.11.7) Consider any surface whose boundary is in the xy-plane. Evaluate

$$\iint(\nabla\times V)\cdot\vec{n}\,d\sigma$$

with $\vec{V}=\langle x-x^2z, yz^3-y^2, x^2y-xz\rangle$.

Solution. The curl of $\vec{V}$ can be computed via

$$\begin{aligned}\operatorname{curl}\vec{V} &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x-x^2z & yz^3-y^2 & x^2y-xz\end{vmatrix}\\ &= (x^2-3z^2y)\vec{i}-(2xy+z+x^2)\vec{j}+0\vec{k}\end{aligned}$$

Now by Stokes' Theorem, if the boundary of our surface is a curve γ lying in the $(xy)-$plane, we have

$$\iint_S(\operatorname{curl}\vec{V})\cdot\vec{n}\,d\sigma=\oint_\gamma\vec{V}\cdot d\vec{r}.$$

On the other hand, the form of the surface is unimportant - all that matters is the boundary! So we can consider a second surface S' which lies in the $(xy)-$plane and whose boundary is also γ:

$$\oint_\gamma\vec{V}\cdot d\vec{r}=\iint_{S'}(\operatorname{curl}\vec{V})\cdot\vec{n}\,d\sigma.$$

In this case, the normal vector is $\vec{n}=\vec{k}$ and is orthogonal to the curl. Therefore, the integral is zero.

(4) (6.11.15) Evaluate the integral

$$\oint_C y\,dx+z\,dy+x\,dz$$

where C is the curve where the plane $x+y=2$ intersects $x^2+y^2+z^2=2(x+y)$.

Solution. Following the idea of problem (3), it's important to choose the surface well when we apply Stokes' Theorem. In this case, the curve is a circle, since it is the intersection of the sphere $(x-1)^2+(y-1)^2+z^2=2$ and the plane $x+y=2$. Since the center $(1,1,0)$ of this sphere lies in the plane, we actually have an equator of the circle. We could choose our surface to be a hemispherical cap, but it is much easier to work with a disk filling in the circle. In that case, we have

$$\oint_C y\,dx + z\,dy + x\,dz = \iint_S (\text{curl}\langle y, z, x\rangle) \cdot d\vec{S} = \iint_S \langle -1, -1, -1\rangle \cdot \vec{n}\,d\sigma.$$

Given the positively oriented circle, the outward unit normal from the plane is $\langle 1, 1, 0\rangle/\sqrt{2}$, and so our integral reduces to

$$\iint_S \langle -1, -1, -1\rangle \cdot \langle 1, 1, 0\rangle/\sqrt{2}\,d\sigma = -\sqrt{2}\iint_S d\sigma.$$

The circle has radius $\sqrt{2}$ (which is the radius of the sphere), and so the integral is just

$$-\sqrt{2}\iint_S d\sigma = -\sqrt{2}(\pi\sqrt{2}^2) = -2\sqrt{2}\pi.$$

For an alternative geometric perspective, note that our curve is also the intersection of the spheres $(x-1)^2 + (y-1)^2 + z^2 = 2$ and $x^2 + y^2 + z^2 = 4$.

(5) (6.11.17b) Show that if Ω is a surface bounded by a curve $\partial\Omega$,

$$\iiint_\Omega \nabla \times \vec{V}\,dV = \iint_{\partial\Omega} \vec{n} \times \vec{V}\,d\sigma.$$

Hint: Apply the divergence theorem to $\vec{V} \times \vec{C}$ with $\vec{C}$ a constant vector. Then make good choices of $\vec{C}$ to compare the coordinates on each side.

Solution. Applying the divergence theorem to the suggested vector field gives us

$$\iiint_\Omega \nabla \cdot (\vec{V} \times \vec{C})\,dV = \iint_{\partial\Omega} (\vec{V} \times \vec{C}) \cdot \vec{n}\,d\sigma.$$

On the right hand side, we have a triple scalar product $(\vec{n}\vec{V}\vec{C})$, which can be reordered as $(\vec{C}\vec{n}\vec{V}) = \vec{C} \cdot (\vec{n} \times \vec{V})$ without changing the sign. Pulling a constant out of the integral, we then have

$$\iint_{\partial\Omega} (\vec{V} \times \vec{C}) \cdot \vec{n}\,d\sigma = \vec{C} \cdot \left(\iint_{\partial\Omega} \vec{n} \times \vec{V}\,d\sigma\right).$$

On the left hand side, we can use the identity (h) from the table of vector identities on page 339 to write

$$\nabla \cdot (\vec{V} \times \vec{C}) = \vec{C} \cdot (\nabla \times \vec{V}) - \vec{V} \cdot (\nabla \times \vec{C}) = \vec{C} \cdot (\nabla \times \vec{V})$$

because the curl of a constant is zero. Therefore, the left hand integral is just

$$\iiint_\Omega \nabla \cdot (\vec{V} \times \vec{C})\,dV = \iiint_\Omega \vec{C} \cdot (\nabla \times \vec{V})\,dV = \vec{C} \cdot \left(\iiint_\Omega \nabla \times \vec{V}\,dV\right).$$

If you'd like to avoid using this identity, then it's easier to have already chosen $\vec{C} = \vec{i}, \vec{j}$, or $\vec{k}$ at this point.

The punchline of this is that for **any** constant vector $\vec{C}$, we have

$$\vec{C} \cdot \left(\iiint_\Omega \nabla \times \vec{V}\,dV\right) = \vec{C} \cdot \left(\iint_{\partial\Omega} \vec{n} \times \vec{V}\,d\sigma\right).$$

Now choose $\vec{C} = \vec{i}, \vec{j}$, or $\vec{k}$. The dot product against a coordinate vector is just the corresponding coordinate, so we immediately see that the $x-$, $y-$, and $z-$coordinates of the two vector integrals are equal; this completes the proof.

Here's another (but much deeper) argument. The key is an idea called *duality*: frequently, to show that two objects are equal what we have to do is show that any function acting on the object gives the same output. Letting $\vec{A}$ and $\vec{B}$ denote the two vector integrals respectively, the question that we really have is

If $\vec{C} \cdot \vec{A} = \vec{C} \cdot \vec{B}$ for all vectors $\vec{C}$, then are $\vec{A}$ and $\vec{B}$ necessarily equal?

The answer is yes as we have seen by *testing* against $\vec{i}, \vec{j}$, and $\vec{k}$. For an even better proof, write $\vec{C} \cdot (\vec{A} - \vec{B}) = 0$ and choose $\vec{C} = \vec{A} - \vec{B}$. It follows that $|\vec{A} - \vec{B}|^2 = 0$, so $\vec{A} = \vec{B}$.

(6) Show that if S is a sphere and $\vec{F}$ is a smooth vector field, then

$$\oiint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = 0.$$

In lecture, we proved something similar by applying the divergence theorem. Here, apply Stokes' theorem: Write the sphere as a union of two hemispheres, and compare the boundaries.

Solution. If we split the sphere into two hemispheres S^+ and S^- above and below the center, the boundary curve of γ each is just an equator of the sphere. One of the boundaries is oriented counterclockwise, while the other is oriented clockwise; this is inherited from the outward normal vector $\vec{n}$ (make sure to draw the picture for this!). Once we have this,

$$\begin{aligned}
\oiint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma &= \iint_{S^+} (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma + \iint_{S^-} (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma \\
&= \oint_{\partial S^+} \vec{F} \cdot d\vec{r} + \oint_{\partial S^-} \vec{F} \cdot d\vec{r} \\
&= \oint_{\gamma} \vec{F} \cdot d\vec{r} + \oint_{-\gamma} \vec{F} \cdot d\vec{r} \\
&= \oint_{\gamma} \vec{F} \cdot d\vec{r} - \oint_{\gamma} \vec{F} \cdot d\vec{r} = 0
\end{aligned}$$

as desired.

An alternative proof using real analysis can also be done as follows; this proof doesn't require any symmetry at all. Punch out a very small hole in the sphere of radius ϵ and get a surface S_ϵ. Then

$$\oiint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = \lim_{\epsilon \to 0^+} \iint_{S_\epsilon} (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = \lim_{\epsilon \to 0^+} \oint_{\partial S_\epsilon} \vec{F} \cdot d\vec{r}.$$

On the other hand, this final term is at most $2\pi\epsilon \max |\vec{F}(x, y, z)|$, which tends to zero as $\epsilon \to 0$.