Math 308 - Exam 1

20 February 2019

Name: Answer key.

Instructions: On the following problems, give full explanations of your answers. Unless otherwise stated, problems are graded on not only the final numerical answer, but also the explanations or work shown. An answer without supporting evidence will receive at most partial credit. No calculators, phones, other electronics, notesheets, or other external aids are allowed on this exam.

Formulas:

- Green's theorem:
$$\oint_{\partial\Omega} P\,dx + Q\,dy = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$
- Divergence theorem:
$$\iiint_{\Omega} \nabla \cdot \vec{F}\,dV = \oiint_{\partial\Omega} \vec{F} \cdot \vec{n}\,d\sigma$$
- Stokes' theorem:
$$\oint_{\partial\Omega} \vec{F} \cdot d\vec{r} = \iint_{\Omega} (\nabla \times \vec{F}) \cdot \vec{n}\,d\sigma$$
- The triple vector product $\vec{A} \times (\vec{B} \times \vec{C})$ is given by
$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}.$$

1. (30 points) Choose three of the following four parts to answer and clearly indicate which parts you want considered. Otherwise, the first three will be graded. All parts are weighted equally.

   (i) Explain why the divergence theorem holds for a very small box B with boundary ∂B (and all sides parallel to the coordinate planes).

   (ii) Explain why the triple scalar product $\vec{a} \cdot (\vec{b} \times \vec{c})$ gives the (signed) volume of the parallelpiped spanned by $\vec{a}, \vec{b}$, and $\vec{c}$ respectively.

   (iii) Suppose a rigid body rotates around the origin with angular velocity $\vec{\omega}$, and that $\vec{r}$ is the position of a point on the object. Explain why the linear velocity $\vec{v}$ satisfies the equation curl $\vec{v} = 2\vec{\omega}$. You may wish to start with the fact that $\vec{v}$ is perpendicular to $\vec{\omega}$ and $\vec{r}$.

   (iv) Suppose you're given a function $T(x, y)$ describing the temperature of a region. Explain how you would determine the direction of heat flow at a point, and why the flow is in this direction.

   Solutions.

   (i) See page 315-316.
   (ii) See page 278-279.
   (iii) See page 324-325.
   (iv) See page 291 for the explanation that the gradient determines the direction of steepest ascent. Then note that heat flows from hot to cold, and therefore the flow is in the direction of steepest descent, which is $-\nabla T$.

2. (60 points) Choose three of the following four parts to answer and clearly indicate which parts you want considered. Otherwise, the first three will be graded. All parts are weighted equally.

   (i) The velocity of a fluid is given by $\vec{V} = \vec{r} + \vec{k}$. Compute the integral

   $$\iint_S \vec{V} \cdot \vec{n}\, d\sigma$$

   where S is the upper half of a sphere of radius 1 centered at the origin, and $\vec{r} = \langle x, y, z \rangle$.

   (ii) Evaluate $\oint \vec{F} \cdot d\vec{r}$ where $\vec{F} = 3y\vec{i} + z\vec{j} + 2x\vec{k}$ and γ is the curve of intersection between the infinite cylinder $x^2 + y^2 = 1$ and the plane $-5x + y + z = 6$.

   (iii) A particle moves in a force field given by $\vec{F} = \langle 2x + 2xy^2, z + 2x^2y, y \rangle$. The particle moves from $(1, 0, 0)$ to $(0, 1, 0)$ by traveling in a circular arc in the xy-plane. Compute the work done on the particle by the field.

   (iv) The temperature in some region of space is given by $T(x, y, z) = x^2 + zy^2$. Compute the rate of change of temperature in the direction $\vec{i} + \vec{j}$ at the point $(1, 2, 3)$.

   Solutions.

   (i) This integral can be computed most simply with the divergence theorem. Close the surface S by adding the disk of radius 1 in the xy-plane; then

   $$\iint_S \vec{V} \cdot \vec{n}\, d\sigma + \iint_D \vec{V} \cdot \vec{n}\, d\sigma = \iiint \nabla \cdot \vec{V}\, dV = \iiint 3\, dV = 3 \cdot \frac{1}{2} \cdot \frac{4}{3}\pi = 2\pi$$

   (since the volume of the hemisphere is half the volume of the sphere).

On the other hand, along D, the normal vector is $-\vec{k}$ and

$$\vec{V} \cdot \vec{n} = (\langle x, y, z \rangle + \langle 0, 0, 1 \rangle) \cdot \langle 0, 0, -1 \rangle = -1.$$

Therefore,

$$\iint_D \vec{V} \cdot \vec{n}\, d\sigma = \iint_D -1\, d\sigma = -\pi.$$

Putting it all together,

$$\iint_S \vec{V} \cdot \vec{n}\, d\sigma = 2\pi + \pi = 3\pi.$$

The integral can also be computed directly. The normal vector at the point $\vec{r}$ is just $\vec{r}$ itself, giving

$$\iint_S \vec{V} \cdot \vec{n}\, d\sigma = \iint_S |r|^2 + \vec{k} \cdot \vec{r}\, d\sigma = \iint_S (1 + z)\, d\sigma.$$

The first half of the integral is just the surface area of the hemisphere, which is 2π. The second half can be done by parametrizing the curve in spherical coordinates via

$$\vec{r}(\theta, \varphi) = \langle \cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi \rangle.$$

We can then compute $d\sigma = |\vec{r}_\theta \times \vec{r}_\varphi|\, d\varphi\, d\theta = \sin\varphi\, d\varphi\, d\theta$ and so

$$\iint_S z\, d\sigma = \int_0^{2\pi} \int_0^{\pi/2} \cos\varphi \sin\varphi\, d\varphi\, d\theta = 2\pi \left[\frac{1}{2}\sin^2\varphi\right]_0^{\pi/2} = \pi.$$

This recovers the same result of 3π as before.

(ii) We will use Stokes' theorem to compute this. Our surface S will be the shape cut out of the cylinder by this plane, and the corresponding normal vector is therefore $-5\vec{i} + \vec{j} + \vec{k}$. The curl of $\vec{F}$ is

$$\nabla \times \vec{F} = (0 - 1)\vec{i} - (2 - 0)\vec{j} + (0 - 3)\vec{k} = -1\vec{i} - 2\vec{j} - 3\vec{k}.$$

Note that $(\nabla \times \vec{F}) \cdot \vec{n} = 5 - 2 - 3 = 0$, so the line integral is

$$\oint \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = 0.$$

(iii) Note that the curl of the vector field is zero, so it is conservative. The corresponding potential is $\varphi = x^2 + x^2y^2 + yz$, in which case $\nabla\varphi = \vec{F}$. Therefore, the fundamental theorem of calculus gives us

$$W = \int_\gamma \vec{F} \cdot d\vec{r} = \varphi(0, 1, 0) - \varphi(1, 0, 0) = -1.$$

(iv) The gradient is $\nabla T = \langle x^2, 2yz, y^2 \rangle = \langle 2, 12, 4 \rangle$ at this point. A unit vector in this direction is $\frac{1}{\sqrt{2}}(\vec{i} + \vec{j})$, so the rate of change is

$$(\nabla T) \cdot \frac{1}{\sqrt{2}}(\vec{i} + \vec{j}) = \frac{14}{\sqrt{2}}.$$