MATH 19620/20

Midterm

NOVEMBER 1, 2018
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Please present your solutions clearly and in an organized way. Answer the questions in the space provided on the question sheets. If you run out of room for an answer, continue on the back of the page. **Please note that the use of a calculator is not allowed.** Good luck!! ◡̈

Full Name: sample solutions

Question	Points	Score
1	20	
2	20	
3	35	
4	20	
5	25	
6	20	
Total:	140	

This exam has 6 questions, for a total of 140 points. The maximum possible score for each problem is given on the right side of the problem.

Please do not open this exam packet unless you are instructed to do so.

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1. Calculate the following. You do not need to show your calculations.

(a) $\begin{bmatrix}1 & 2\\3 & 1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix}$ $\boxed{5}$

(b) $\begin{bmatrix}1 & 2\\3 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\0 & 2 & 0\end{bmatrix} = \begin{bmatrix}0 & 5 & 0\\0 & 5 & 0\end{bmatrix}$ $\boxed{5}$

(c) $\begin{bmatrix}2 & 1\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}10\end{bmatrix}$ $\boxed{5}$

(d) $\begin{bmatrix}3\\4\end{bmatrix}\begin{bmatrix}2 & 1\end{bmatrix} = \begin{bmatrix}6 & 3\\8 & 4\end{bmatrix}$ $\boxed{5}$

2. Let $A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 3 & 1\end{bmatrix}$.

(a) Is A invertible? If so, what is A^{-1}? (Hint: You should be able to fit the calculations in the space provided.) $\boxed{10}$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 0 & 1 & 0\\0 & 3 & 1 & 0 & 0 & 1\end{array}\right] -3(\mathrm{II}) \longrightarrow \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 0 & 1 & 0\\0 & 0 & 1 & 0 & -3 & 1\end{array}\right]$$

A is invertible, $A^{-1} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -3 & 1\end{bmatrix}$

(b) What is im(A)? $\boxed{5}$

A is invertible $\Longrightarrow$ $\mathrm{im}(A) = \mathbb{R}^3$

(c) What is ker(A)? $\boxed{5}$

A is invertible $\Longrightarrow$ $\ker(A) = \{\vec{0}\}$

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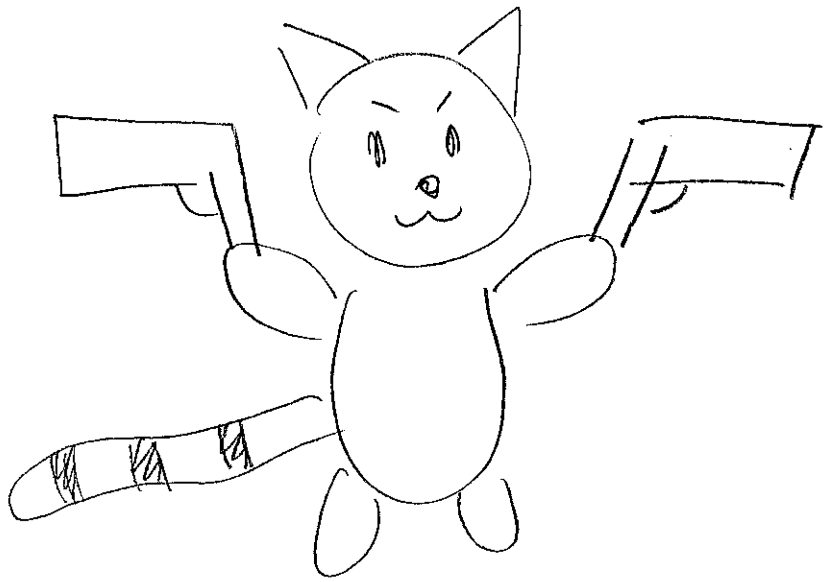
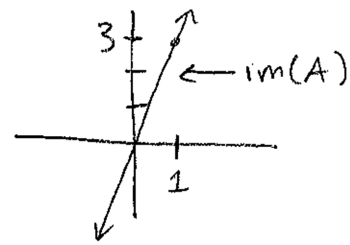


3. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \end{bmatrix}$.

(a) What is im(A)? There are many acceptable ways to answer this (e.g., with a picture or with a geometric description). **10**

All 3 columns are parallel, so

$$\operatorname{im} A = \operatorname{span}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \left\{ \begin{bmatrix} t \\ 3t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$
$$= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, y = 3x \right\}$$



(b) Find rref(A). **5**

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \end{bmatrix} \begin{matrix} \\ -3(I) \end{matrix} \longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{rref}(A)$$

(c) What are all the solutions to the following system? **10**

$$\begin{aligned} x + 3y + 2z &= 3 \\ 3x + 9y + 6z &= -1 \end{aligned}$$

$\begin{bmatrix} 3 \\ -1 \end{bmatrix} \notin \operatorname{im}(A)$ so there are no solutions

(d) What are all the solutions to the following system? **10**

$$\begin{aligned} x + 3y + 2z &= 0 \\ 3x + 9y + 6z &= 0 \end{aligned}$$

The second equation is a multiple of the first, so the solutions are

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \,\middle|\, x + 3y + 2z = 0 \right\} = \left\{ \begin{bmatrix} -3y - 2z \\ y \\ z \end{bmatrix} \,\middle|\, y, z \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left(\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right)$$

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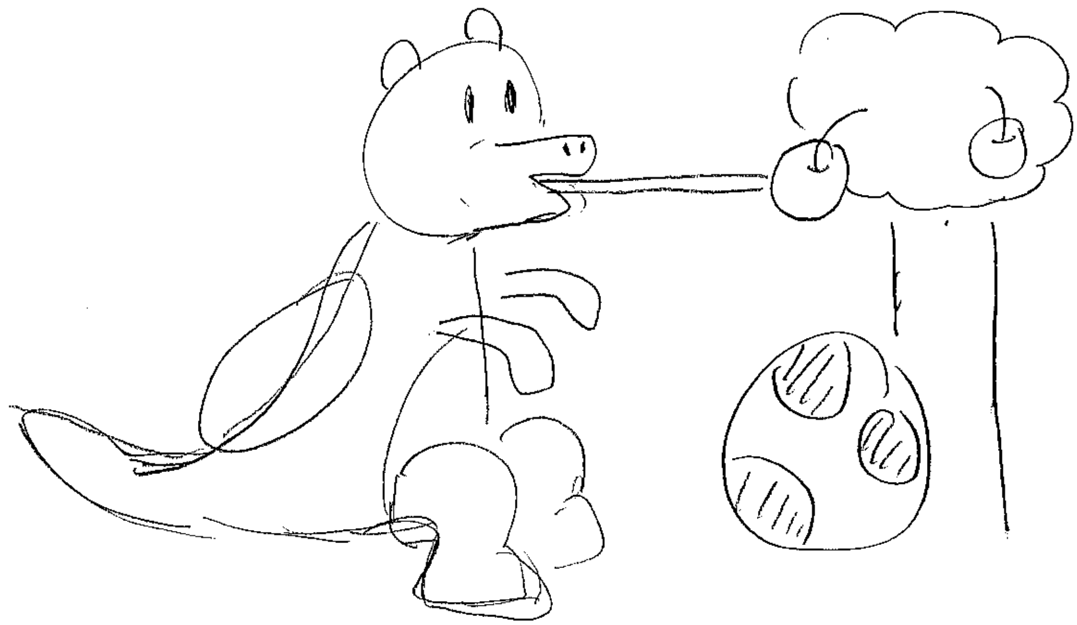


4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

(a) Give a geometric description of this transformation. $\boxed{10}$

T scales every vector by 5.

(b) Give a geometric description of the inverse T^{-1}. $\boxed{5}$

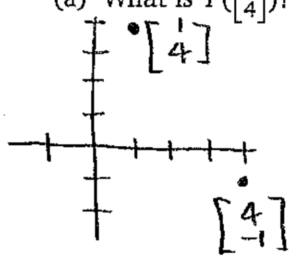
T^{-1} scales every vector by $1/5$.

(c) What is the matrix A^{-1}? $\boxed{5}$

Using (b), we know $A^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates points **clockwise** by 90 degrees around the origin.

(a) What is $T(\begin{bmatrix} 1 \\ 4 \end{bmatrix})$? (Maybe drawing a picture would help.) $\boxed{5}$



$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

(b) What is the matrix A for which $T(\vec{x}) = A\vec{x}$? $\boxed{10}$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(c) What is A^{40}? $\boxed{10}$

$A^4 = I_2$ since rotating $90°$ clockwise 4 times is the same as doing nothing.

$$\Longrightarrow A^{40} = (A^4)^{10} = I_2^{10} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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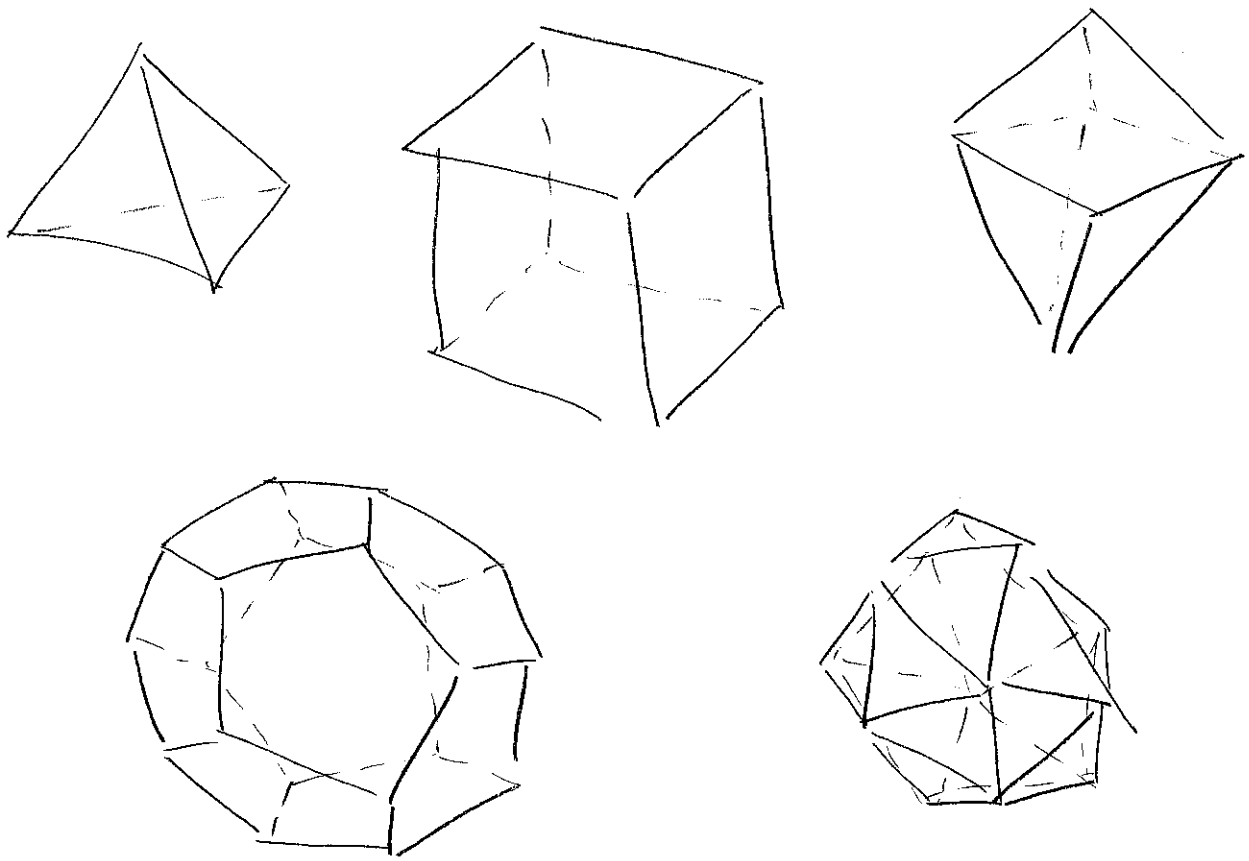

6. Determine if the following functions are linear transformations or not.

(a) $T : \mathbb{R}^1 \to \mathbb{R}^1$, $T(x) = 4x - 2$ $\boxed{10}$

$T(0) = -2 \neq 0$. Not linear.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^4$, $T(\vec{x}) = \vec{0}$. (In other words, $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.) $\boxed{10}$

T is linear.

Solution 1:

$$T(\vec{x}+\vec{y}) = \vec{0}$$
$$T(\vec{x}) + T(\vec{y}) = \vec{0} + \vec{0} = \vec{0}$$
equal

$$T(k\vec{x}) = \vec{0}$$
$$k\,T(\vec{x}) = k\vec{0} = \vec{0}$$
equal

Solution 2:

$$T(\vec{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{x}$$

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Me too...