MATH 19620/20

Final exam

DECEMBER 11, 2018
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Please present your solutions clearly and in an organized way. Answer the questions in the space provided on the question sheets. If you run out of room for an answer, continue on the back of the page. **Please note that the use of a calculator is not allowed.** Good luck!! ◡̈

Full Name: Sample solutions

Question	Points	Score
1	39	
2	10	
3	16	
4	16	
5	15	
6	25	
7	43	
8	25	
9	15	
10	4	
Total:	208	

This exam has 10 questions, for a total of 208 points. The maximum possible score for each problem is given on the right side of the problem.

Please do not open this exam packet unless you are instructed to do so.

Here are some formulas:

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \qquad A^TA\vec{x} = A^T\vec{b}, \qquad QQ^T, \qquad A(A^TA)^{-1}A^T, \qquad A = SBS^{-1}$$

1. Consider the matrix

$$A = \begin{bmatrix} 4 & 1 & 1 & 7 & 6 \\ 3 & 1 & 2 & 6 & 5 \\ 2 & 1 & 9 & 11 & 10 \\ 1 & 3 & 2 & -2 & 0 \end{bmatrix}, \text{ which satisfies } \operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5$ be the columns of A (in that order).
Let $\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5$ be the columns of rref(A) (in that order).

(a) Using rref(A) to help you, find a basis for ker(A). **8**

$$\begin{aligned} x_1 + 2x_4 &= 0 \\ x_2 - 2x_4 &= 0 \\ x_3 + x_4 &= 0 \\ x_5 &= 0 \end{aligned} \implies \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_4 \\ 2x_4 \\ -x_4 \\ x_4 \\ 0 \end{bmatrix}$$

$$\implies \ker(A) = \operatorname{span}\left(\begin{bmatrix} -2 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right)$$

(b) Using rref(A) to help you, find a basis for im(A). **8**

rref(A) has pivots in columns 1, 2, 3, 5

$$\implies \operatorname{im}(A) = \operatorname{span}(\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_5).$$

Question continued on next page.

(c) Fill in the four blanks. (You do not need to justify.) $\boxed{8}$

- The kernel of A is a <u>1</u>-dimensional subspace of $\mathbb{R}^p$, where $p =$ <u>5</u>.
- The image of A is a <u>4</u>-dimensional subspace of $\mathbb{R}^q$, where $q =$ <u>4</u>.

(d) Instead of writing im(A) as the span of some vectors, what is an easier way to describe im(A)? Please explain briefly. *Hint:* Use part (c). $\boxed{5}$

The only 4-dimensional subspace of $\mathbb{R}^4$ is $\mathbb{R}^4$ itself $\Longrightarrow$ $\text{im}(A) = \mathbb{R}^4$

(e) Find a nontrivial relation among $\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5$. (You do not need to justify.) $\boxed{5}$

$$-2\vec{b}_1 + 2\vec{b}_2 - \vec{b}_3 + \vec{b}_4 = \vec{0}$$

(f) Find a nontrivial relation among $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5$. (You do not need to justify.) $\boxed{5}$

$$-2\vec{a}_1 + 2\vec{a}_2 - \vec{a}_3 + \vec{a}_4 = \vec{0}$$

2. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}.$$

Fill in the following blanks. (You do not need to justify.)

(a) The equation $A\vec{x} = \vec{b}$ is a system of <u>3</u> equations in <u>2</u> variables. $\boxed{4}$

(b) The **normal equation** of $A\vec{x} = \vec{b}$ is a system of <u>2</u> equations in <u>2</u> variables. $\boxed{6}$

3. Let $A = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

(a) Find the least-squares solution of the system $A\vec{x} = \vec{b}$. 10

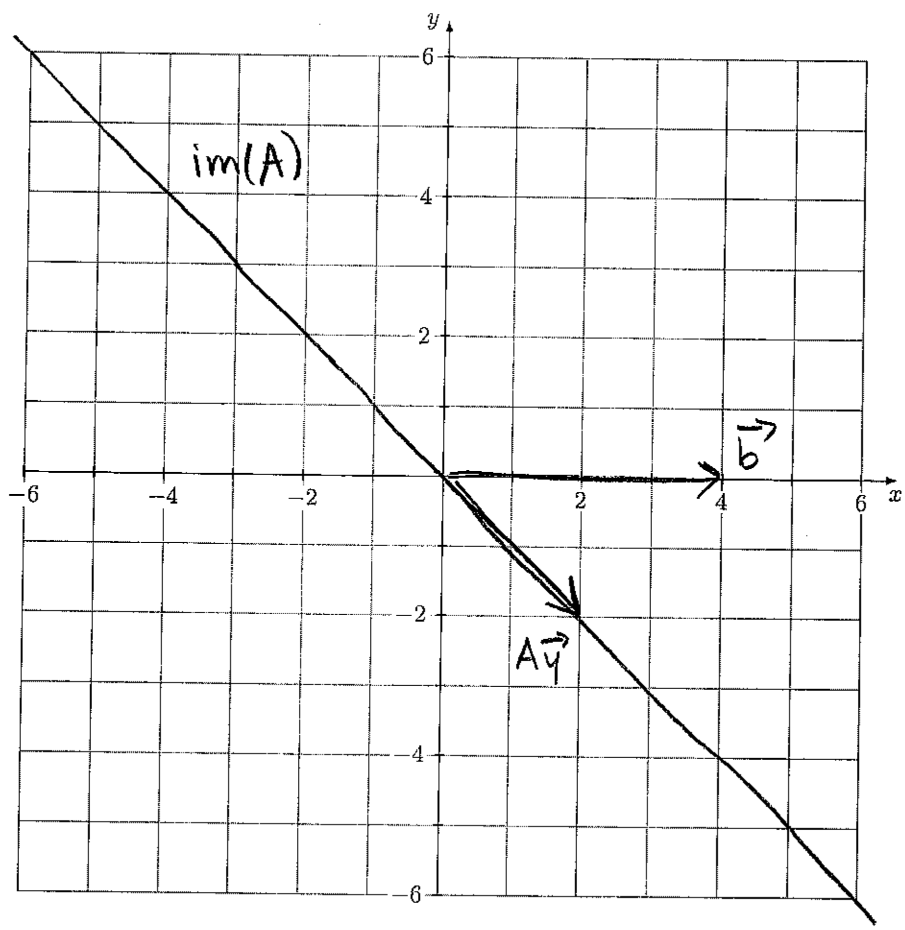
$$A^TA = [-2 \;\; 2]\begin{bmatrix} -2 \\ 2 \end{bmatrix} = 8 \qquad A^T\vec{b} = [-2 \;\; 2]\begin{bmatrix} 4 \\ 0 \end{bmatrix} = -8$$

$$A^TA\vec{x}^* = A^T\vec{b}$$

$$8\vec{x}^* = -8$$

$$\boxed{\vec{x}^* = -1}$$

(b) Let $\vec{y}$ be the solution to part (a). Draw a sketch showing the vector $\vec{b}$, the subspace im(A), and the vector $A\vec{y}$. Please label each one. 6



4. Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Verify that these three vectors are orthogonal to each other. Please show your calculations. $\boxed{6}$

$$\vec{v}_1 \cdot \vec{v}_2 = 1 + (-2) + 1 = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = 1 + 0 + (-1) = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = 1 + 0 + (-1) = 0$$

(b) Are these vectors orthonormal? Please explain. $\boxed{5}$

No. $\vec{v}_1 \cdot \vec{v}_1 = 1 + 4 + 1 = 6 \Rightarrow \vec{v}_1$ is not a unit vector

(c) Let $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Suppose $\underbrace{\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3}$. Find c_1. *Hint:* Use part (a). $\boxed{5}$

Dot both sides with $\vec{v}_1$.

$$\vec{x} \cdot \vec{v}_1 = c_1 \vec{v}_1 \cdot \vec{v}_1 + c_2 \vec{v}_2 \cdot \vec{v}_1 + c_3 \vec{v}_3 \cdot \vec{v}_1$$

$$1 = 6c_1$$

$$\boxed{1/6 = c_1}$$

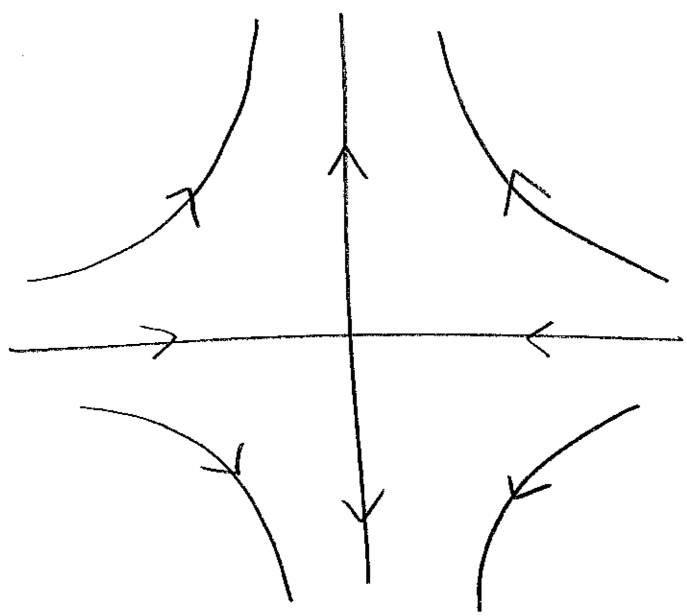
5. Consider the matrix

$$A = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.1 \end{bmatrix}$$

(a) Find an eigenbasis for A. (You do not need to justify.) $\boxed{5}$

$$\left.\begin{aligned} A\vec{e_1} &= 0.9\vec{e_1} \\ A\vec{e_2} &= 1.1\vec{e_2} \end{aligned}\right\} \Longrightarrow \vec{e_1}, \vec{e_2} \text{ is an eigenbasis for } A$$

(b) Sketch a phase portrait for A. $\boxed{10}$

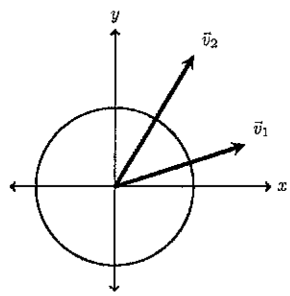


6. Suppose we want to perform Gram-Schmidt.
We start with two linearly independent vectors $\vec{v}_1, \vec{v}_2$ and end with two orthonormal vectors $\vec{u}_1, \vec{u}_2$.

(a) Here are the calculations we need for Gram-Schmidt. Please fill in the one for $\vec{u}_2$. $\boxed{5}$

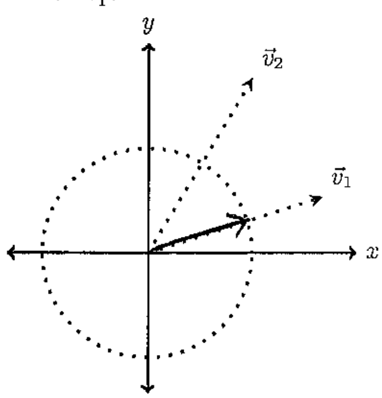
$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1, \qquad \vec{v}_2^{\|} = (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1, \qquad \vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\|}, \qquad \vec{u}_2 = \frac{1}{\| \vec{v}_2^{\perp} \|} \vec{v}_2^{\perp}$$

(b) Suppose $\vec{v}_1, \vec{v}_2$ are as in the diagram below. The unit circle $x^2 + y^2 = 1$ is also displayed on the diagram. $\boxed{20}$

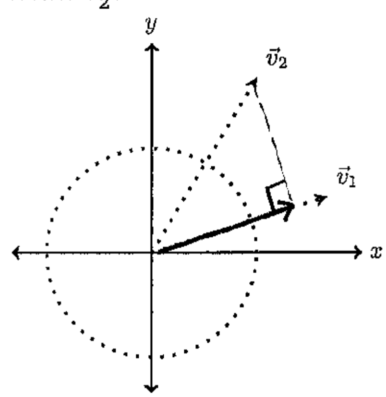


Please draw the following vectors. They do not need to be precise, but should be fairly accurate. (The arrows should all begin at the origin.)

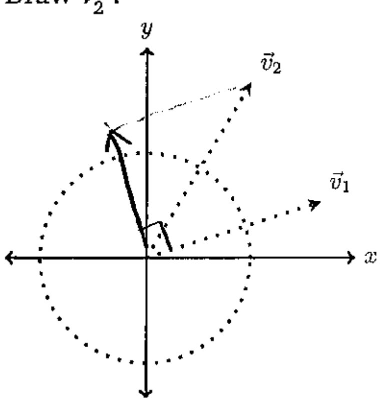
Draw $\vec{u}_1$:



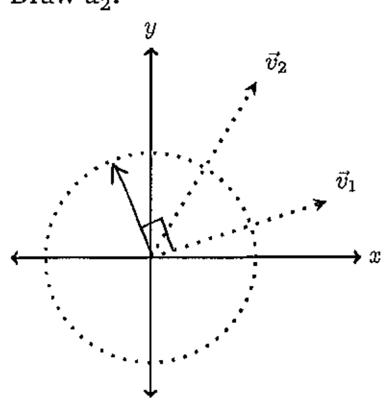
Draw $\vec{v}_2^{\|}$:



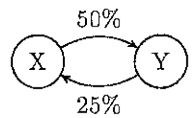
Draw $\vec{v}_2^{\perp}$:



Draw $\vec{u}_2$:



7. Consider the following state diagram:



(a) Write down the transition matrix for this Markov chain. Call your matrix A. $\boxed{5}$

$$A = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$$

(b) Find the eigenvalues of A. (It may be easier to calculate with fractions instead of decimals.) $\boxed{10}$

$$\det(A-\lambda I) = \det \begin{bmatrix} 1/2-\lambda & 1/4 \\ 1/2 & 3/4-\lambda \end{bmatrix} = \left(\tfrac{1}{2}-\lambda\right)\left(\tfrac{3}{4}-\lambda\right) - \tfrac{1}{8}$$

$$= \lambda^2 - \tfrac{5}{4}\lambda + \tfrac{3}{8} - \tfrac{1}{8} = \lambda^2 - \tfrac{5}{4}\lambda + \tfrac{1}{4} = (\lambda - 1)(\lambda - \tfrac{1}{4})$$

$\Longrightarrow$ eigenvalues are 1 and $1/4$.

(c) Find a basis of each eigenspace of A. $\boxed{10}$

$$E_1: \quad \ker(A-I) = \ker \begin{bmatrix} -1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$E_{1/4}: \quad \ker(A-\tfrac{1}{4}I) = \ker \begin{bmatrix} 1/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Question continued on next page.

(d) Find matrices S and B such that B is diagonal and $A = SBS^{-1}$. (Just write down the matrices. You do not need to justify.) 5

$$S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}$$

(e) Let $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find $S^{-1}\vec{x}_0$. 2

$$S^{-1} = \frac{1}{-1-2}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$S^{-1}\vec{x}_0 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

(f) Find a closed formula for $A^t\vec{x}_0$. 8

$$A^t\vec{x}_0 = SB^tS^{-1}\vec{x}_0 = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (1/4)^t \end{bmatrix}\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1/3 \\ 2/3\,(1/4)^t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} + \frac{2}{3}\left(\frac{1}{4}\right)^t \\ \frac{2}{3} - \frac{2}{3}\left(\frac{1}{4}\right)^t \end{bmatrix}$$

(g) Find $\lim_{t\to\infty} A^t\vec{x}_0$. 3

$$\lim_{t\to\infty}\left(\frac{1}{4}\right)^t = 0 \quad \text{so} \quad \lim_{t\to\infty} A^t\vec{x}_0 = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

8. Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Find the eigenvalues of A. 10

$$\begin{bmatrix} 1-\lambda & -2 & 0 \\ -1 & 2-\lambda & -2 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{matrix} 1-\lambda & -2 \\ -1 & 2-\lambda \\ 0 & 1-\lambda \end{matrix}$$

$$\det(A-\lambda I) = (1-\lambda)(2-\lambda)(1-\lambda) - (1-\lambda)(-1)(-2)$$

$$= (1-\lambda)\left[(2-\lambda)(1-\lambda) - (-1)(-2)\right]$$

$$= (1-\lambda)(\lambda^2 - 3\lambda + 2 - 2)$$

$$= (1-\lambda)(\lambda^2 - 3\lambda)$$

$$= (1-\lambda)(\lambda)(\lambda-3)$$

$\Longrightarrow$ eigenvalues are $0, 1, 3$

(b) Is A invertible? Please explain. 5

0 is an eigenvalue $\Rightarrow$ $\det A = 0$

$\Rightarrow$ A is not invertible.

(c) Let $\vec{h} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Compute $A\vec{h}$. 3

$$\begin{bmatrix} 1 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2 \\ -1-2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$$

(d) Compute $A^{100}\vec{h}$. 7

$$A\vec{h} = 3\vec{h} \Longrightarrow A^{100}\vec{h} = 3^{100}\vec{h} = \begin{bmatrix} 3^{100} \\ -3^{100} \\ 0 \end{bmatrix}$$

9. In $\mathbb{R}^3$ consider the subspace V spanned by $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.

(a) Find the matrix of the orthogonal projection onto V. $\boxed{10}$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{v}}{\sqrt{1+4+4}} = \frac{1}{3}\vec{v}$$

$$QQ^T = \frac{1}{9}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix}$$

(b) Let $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Find the orthogonal projection of $\vec{x}$ onto V. $\boxed{5}$

$$QQ^T\vec{x} = \frac{1}{9}\begin{bmatrix} -2 \\ -4 \\ 4 \end{bmatrix}$$

10. (a) Of all the material that we covered in this class, what was your favorite? $\boxed{2}$

The coyote and roadrunner dynamical system

(b) Least favorite? $\boxed{2}$

You guys said my  looks like a fox.
That made me sad. Just kidding. :)