

Please present your solutions clearly and in an organized way. Answer the questions in the space provided on the question sheets. If you run out of room for an answer, continue on the back of the page. **Please note that the use of a calculator is not allowed.** Good luck!! 😊

Full Name: _____

Question	Points	Score
1	39	
2	10	
3	16	
4	16	
5	15	
6	25	
7	43	
8	25	
9	15	
10	4	
Total:	208	

This exam has 10 questions, for a total of 208 points. The maximum possible score for each problem is given on the right side of the problem.



Here are some formulas:

$$\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad A^T A \vec{x} = A^T \vec{b}, \quad QQ^T, \quad A(A^T A)^{-1} A^T, \quad A = SBS^{-1}$$

1. Consider the matrix

$$A = \begin{bmatrix} 4 & 1 & 1 & 7 & 6 \\ 3 & 1 & 2 & 6 & 5 \\ 2 & 1 & 9 & 11 & 10 \\ 1 & 3 & 2 & -2 & 0 \end{bmatrix}, \text{ which satisfies } \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5$ be the columns of A (in that order).

Let $\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5$ be the columns of $\text{rref}(A)$ (in that order).

(a) Using $\text{rref}(A)$ to help you, find a basis for $\ker(A)$.

8

(b) Using $\text{rref}(A)$ to help you, find a basis for $\text{im}(A)$.

8

Question continued on next page.



(c) Fill in the four blanks. (You do not need to justify.)

8

- The kernel of A is a _____-dimensional subspace of \mathbb{R}^p , where $p = \underline{\hspace{1cm}}$.
- The image of A is a _____-dimensional subspace of \mathbb{R}^q , where $q = \underline{\hspace{1cm}}$.

(d) Instead of writing $\text{im}(A)$ as the span of some vectors, what is an easier way to describe $\text{im}(A)$? Please explain briefly. *Hint:* Use part (c).

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(e) Find a nontrivial relation among $\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5$. (You do not need to justify.)

5

(f) Find a nontrivial relation among $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5$. (You do not need to justify.)

5

2. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}.$$

Fill in the following blanks. (You do not need to justify.)

(a) The equation $A\vec{x} = \vec{b}$ is a system of _____ equations in _____ variables.

4

(b) The **normal equation** of $A\vec{x} = \vec{b}$ is a system of _____ equations in _____ variables.

6



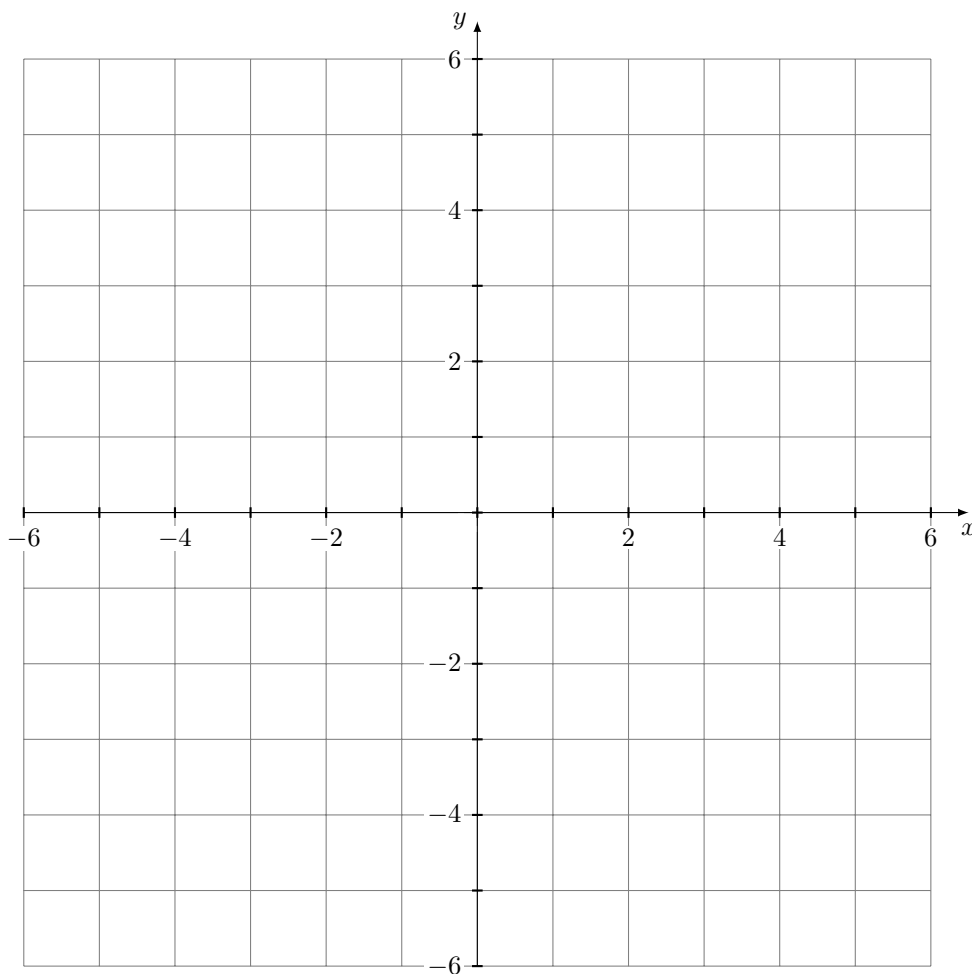
3. Let $A = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

(a) Find the least-squares solution of the system $A\vec{x} = \vec{b}$.

10

(b) Let \vec{y} be the solution to part (a). Draw a sketch showing the vector \vec{b} , the subspace $\text{im}(A)$, and the vector $A\vec{y}$. Please label each one.

6





4. Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Verify that these three vectors are orthogonal to each other. Please show your calculations.

6

(b) Are these vectors orthonormal? Please explain.

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(c) Let $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Suppose $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$. Find c_1 . *Hint:* Use part (a).

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5. Consider the matrix

$$A = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.1 \end{bmatrix}$$

(a) Find an eigenbasis for A . (You do not need to justify.)

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(b) Sketch a phase portrait for A .

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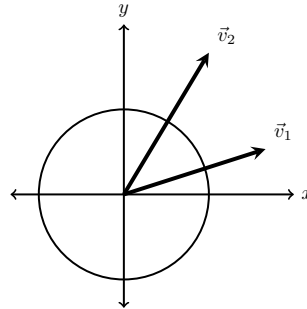
6. Suppose we want to perform Gram-Schmidt.

We start with two linearly independent vectors \vec{v}_1, \vec{v}_2 and end with two orthonormal vectors \vec{u}_1, \vec{u}_2 .

(a) Here are the calculations we need for Gram-Schmidt. Please fill in the one for \vec{u}_2 . 5

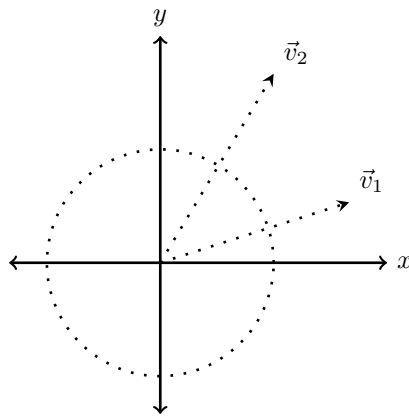
$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1, \quad \vec{v}_2^{\parallel} = (\vec{u}_1 \cdot \vec{v}_2) \vec{u}_1, \quad \vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel}, \quad \vec{u}_2 =$$

(b) Suppose \vec{v}_1, \vec{v}_2 are as in the diagram below. The unit circle $x^2 + y^2 = 1$ is also displayed on the diagram. 20

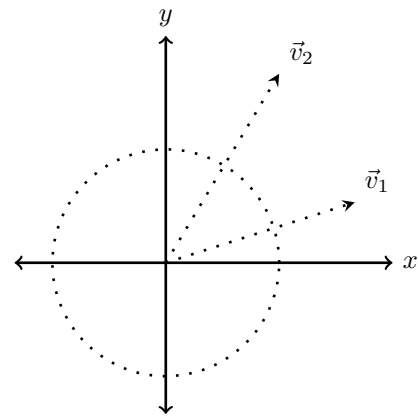


Please draw the following vectors. They do not need to be precise, but should be fairly accurate. (The arrows should all begin at the origin.)

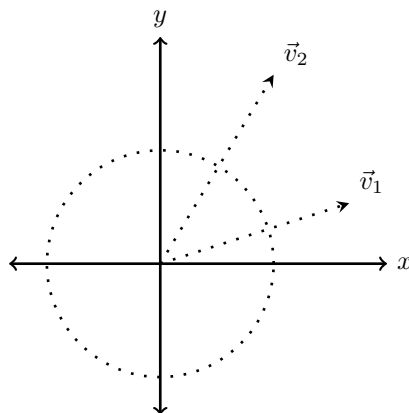
Draw \vec{u}_1 :



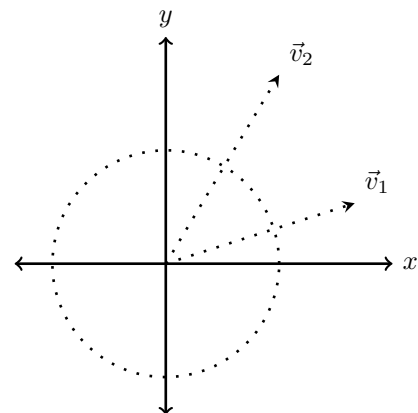
Draw \vec{v}_2^{\parallel} :



Draw \vec{v}_2^{\perp} :

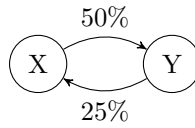


Draw \vec{u}_2 :





7. Consider the following state diagram:



(a) Write down the transition matrix for this Markov chain. Call your matrix A .

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(b) Find the eigenvalues of A . (It may be easier to calculate with fractions instead of decimals.)

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(c) Find a basis of each eigenspace of A .

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Question continued on next page.



- (d) Find matrices S and B such that B is diagonal and $A = SBS^{-1}$.
(Just write down the matrices. You do not need to justify.)

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- (e) Let $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find $S^{-1}\vec{x}_0$.

2

- (f) Find a closed formula for $A^t \vec{x}_0$.

8

- (g) Find $\lim_{t \rightarrow \infty} A^t \vec{x}_0$.

3



8. Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Find the eigenvalues of A .

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(b) Is A invertible? Please explain.

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(c) Let $\vec{h} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Compute $A\vec{h}$.

3

(d) Compute $A^{100}\vec{h}$.

7



9. In \mathbb{R}^3 consider the subspace V spanned by $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.

(a) Find the matrix of the orthogonal projection onto V .

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(b) Let $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Find the orthogonal projection of \vec{x} onto V .

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10. (a) Of all the material that we covered in this class, what was your favorite?

2

(b) Least favorite?

2