Midterm #2

FEBRUARY 20, 2020
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Please present your solutions clearly and in an organized way. Simplify all your final answers. If an answer box is given, write your final answer in the box. If you run out of room, continue on the extra pages provided at the end. **The use of a calculator is not allowed.** Good luck!! ‿

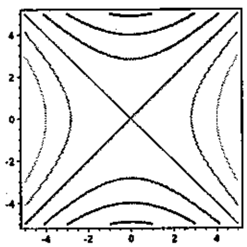
Full Name: Sample solutions

Student ID:

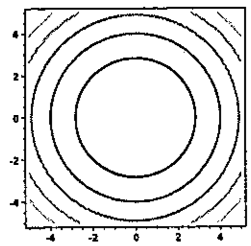
Question	Points	Score
1	9	
2	10	
3	20	
4	20	
5	22	
6	10	
7	10	
Total:	101	

This exam has 7 questions, for a total of 101 points. The maximum possible score for each problem is given on the right side of the problem.

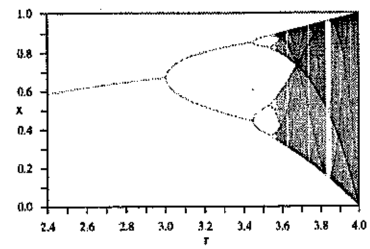
1. Match the plots to the descriptions below by writing the appropriate letter (A, B, C) in the boxes. (No justification needed.) 9



(A)



(B)



(C)

1. The bifurcation diagram for the logistic map $\boxed{C}$

2. The level curves of the function $f(x,y) = x^2 + y^2$ $\boxed{B}$

3. The level curves of the function $f(x,y) = x^2 - y^2$ $\boxed{A}$

2. Find an equation for the plane which passes through (3,2,0) and is perpendicular to the vector (2,-1,1). Write your final answer in the form $Ax + By + Cz = D$, where A, B, C, D are numbers. 10

equation of plane: $\boxed{2x - y + z = 4}$

$$(2,-1,1)\cdot((x,y,z)-(3,2,0)) = 0.$$

$$(2,-1,1)\cdot(x,y,z) - (2,-1,1)\cdot(3,2,0) = 0.$$

$$2x - y + z - 4 = 0$$

$$2x - y + z = 4$$

3. Determine whether the series converges or diverges. Please justify.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{1/10}}$ circle one: CONVERGES DIVERGES $\boxed{10}$

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k \qquad a_k = \frac{1}{k^{1/10}}$$

- a_k is positive
- a_k is a decreasing sequence
- $a_k \longrightarrow 0$

So $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges by the alternating series test.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{3k+1}{\sqrt{k^4+k+1}}$ circle one: CONVERGES DIVERGES $\boxed{10}$

Let $a_k = \dfrac{3k+1}{\sqrt{k^4+k+1}}$ $\qquad b_k = \dfrac{1}{k}$

$$\frac{a_k}{b_k} = \frac{3k+1}{\sqrt{k^4+k+1}} \cdot \frac{k}{1} = \frac{3k^2+k}{\sqrt{k^4+k+1}} \cdot \frac{\frac{1}{k^2}}{\frac{1}{k^2}} = \frac{3k^2+k}{\sqrt{k^4+k+1}} \cdot \frac{\frac{1}{k^2}}{\frac{1}{\sqrt{k^4}}}$$

$$= \frac{3+\frac{1}{k}}{\sqrt{1+\frac{1}{k^3}+\frac{1}{k^4}}}$$

So $\displaystyle\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{3+0}{\sqrt{1+0+0}} = 3$

Since $\sum b_k$ diverges, $\sum a_k$ diverges by the limit comparison test

4. If possible, give an example of a series satisfying the given properties. If it is impossible, no explanation is needed.

(a) $\sum a_k$ converges and $\sum |a_k|$ converges. circle one: POSSIBLE IMPOSSIBLE $\boxed{5}$

$a_k = 0$ $|a_k| = 0$

$\sum 0$ converges $\sum 0$ converges

(b) $\sum a_k$ converges and $\sum |a_k|$ diverges. circle one: POSSIBLE IMPOSSIBLE $\boxed{5}$

$a_k = \frac{(-1)^{k+1}}{k}$ $|a_k| = \frac{1}{k}$

$\sum \frac{(-1)^{k+1}}{k}$ converges (alternating series test) $\sum \frac{1}{k}$ diverges (integral test)

(c) $\sum a_k$ diverges and $\sum |a_k|$ converges. circle one: POSSIBLE IMPOSSIBLE $\boxed{5}$

By the absolute convergence test, if $\sum |a_k|$ converges, then $\sum a_k$ converges.

(d) $\sum a_k$ diverges and $\sum |a_k|$ diverges. circle one: POSSIBLE IMPOSSIBLE $\boxed{5}$

$a_k = 1$ $|a_k| = 1$

$\sum 1$ diverges $\sum 1$ diverges

5. (a) For the following four statements, circle either TRUE or FALSE. No explanation is needed. (Hint: Exactly two are TRUE.) $\boxed{12}$

If $a_k \to 0$, then $\sum a_k$ converges.	TRUE	FALSE
If $\sum a_k$ diverges, then $a_k \not\to 0$.	TRUE	FALSE
If $a_k \not\to 0$, then $\sum a_k$ diverges.	TRUE	FALSE
If $\sum a_k$ converges, then $a_k \to 0$.	TRUE	FALSE

(b) Show that if $\sum a_k$ converges, then $\sum \dfrac{(a_k+1)^{15300}}{a_k^2+2020}$ diverges. (Hint: Use part (a).) $\boxed{10}$

If $\sum a_k$ converges,

then $\lim_{k\to\infty} a_k = 0$

So $\displaystyle\lim_{k\to\infty} \frac{(a_k+1)^{15300}}{a_k^2+2020} = \frac{(0+1)^{15300}}{0+2020} = \frac{1}{2020} \neq 0$

So $\displaystyle\sum \frac{(a_k+1)^{15300}}{a_k^2+2020}$ diverges.

10

6. Determine whether the lines ℓ_1 and ℓ_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

ℓ_1: $x_1(t) = 2 + t$, $y_1(t) = -1 - t$, $z_1(t) = 3 + 2t$

ℓ_2: $x_2(u) = 1 + u$, $y_2(u) = 1 - 2u$, $z_2(u) = 1 + 2u$

direction vector for ℓ_1: $(1, -1, 2)$

for ℓ_2: $(1, -2, 2)$.

so the lines are not parallel.

To check for intersection:

$$\begin{cases} ① \; 2+t = 1+u \\ ② \; -1+t = 1-2u \\ ③ \; 3+2t = 1+2u \end{cases} \longrightarrow \begin{cases} t = -1+u \\ t = 2-2u \end{cases} \longrightarrow \begin{aligned} -1+u &= 2-2u \\ 3u &= 3 \\ u &= 1 \end{aligned}$$

$t = -1+1 = 0$.

we know $t=0$, $u=1$ satisfies ① and ②.

To check ③: $3+2(0) = 1+2(1)$

$3 = 3$ ✓

So the two lines intersect

$x_1(0) = 2$

$y_1(0) = -1$

$z_1(0) = 3$

point of intersection is $(2, -1, 3)$.

10

7. Suppose $\vec{a}$ and $\vec{b}$ are two perpendicular vectors. Using vector operations, prove the following:

$$\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2$$

(Hint: Start by expanding the left-hand side.)

(Fun fact: This is a way of proving the Pythagorean theorem.)

$$\|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$

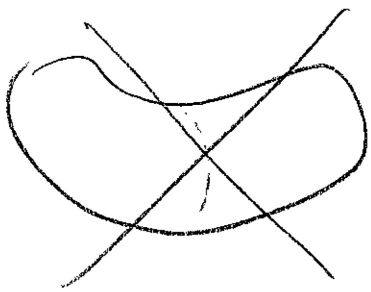
$$= \vec{a}\cdot\vec{a} + \underbrace{\vec{a}\cdot\vec{b}} + \underbrace{\vec{b}\cdot\vec{a}} + \vec{b}\cdot\vec{b}$$

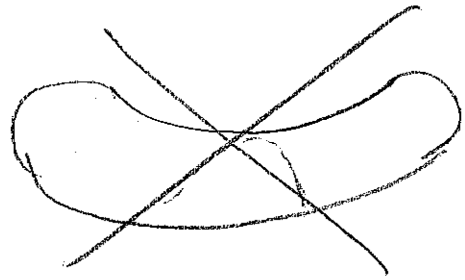
these are zero since $\vec{a} \perp \vec{b}$

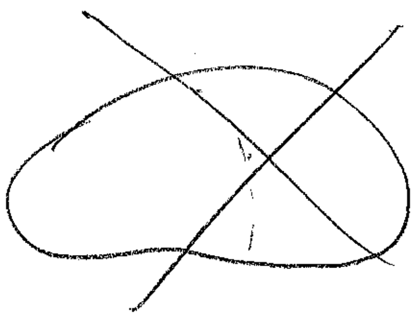
$$= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= \|\vec{a}\|^2 + \|\vec{b}\|^2$$

This is blank space. If you are bored, try to draw a Pringle®.







~~I can't do it.~~

I just need some more practice.